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# DIFFERENTIAL AND INTEGRAL CALCULUS

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# ELEMENTS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS

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# PREFACE

The present volume is the result of an effort to write a modern text-book on the Calculus which shall be essentially a drill book. With this end in view, the pedagogic principle that each result should be made intuitionally as well as analytically evident to the student has been kept constantly in mind. Indeed it has been thought best in some cases, as for example in Maxima and Minima and the Theorem of Mean Value, to discuss the question first from the intuitional side, in order that the significance of the new idea might be made plain in the most direct manner. The object has not been to teach the student to rely upon his intuition, but in some cases to use this faculty in advance of the analytical investigation. The short chapter on Numbers is intended to give the student a chance to review his ideas of number. Limits and Continuity are treated at length, the latter mostly from a graphical standpoint, — the only method suited to a first course. In fact, graphical illustration has been drawn upon to the fullest extent throughout the book.

As special features, attention may be called to the effort to make perfectly clear the nature and extent of each new theorem, the large number of carefully graded exercises, and the summarizing into working rules of the methods of solving problems. In the Integral Calculus the notion of integration over a plane area has been much enlarged upon, and integration as the limit of a summation is constantly emphasized. The book contains more material than is necessary for the usual course of one hundred lessons given in our colleges and engineering schools; but this gives teachers an opportunity to choose such topics as best suit the needs of their classes. It is believed that the volume contains all subjects from which a selection naturally would be made in preparing students either for elementary work in applied science or for more advanced work in pure mathematics.

Certain proofs of considerable difficulty (as the existence of the number *e*) have been inserted with the belief that, while it is not always advisable to require beginners to learn them, a discussion of them with the class will render such investigations profitable and stimulating.

With a few exceptions the author has found it impracticable to acknowledge his indebtedness to the large number of American, English, and continental writers whose books and articles have helped and inspired him in the work, the bulk of the matter having long been the common property of all mankind. While many of the exercises are new, a large number are standard and are to be found in many of the best treatises.

The author's acknowledgments are due to Professor M. B. Porter of the University of Texas for critically examining the manuscript, to Professor James Pierpont of Yale University for many valuable suggestions, to my former colleagues, Professor E. R. Hedrick of the University of Missouri and Dr. C. N. Haskins, for their interest and assistance, to Dr. C. E. Stromquist of the University of Princeton for verifying the examples, and to my colleague, Mr. L. C. Weeks, for drawing the figures. The thanks of the author are also due to his former instructor in mathematics, Professor John E. Clark, now Professor Emeritus of Yale University, who first advised and encouraged him to undertake the task of writing this book.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY, July, 1904.

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# DIFFERENTIAL CALCULUS

## CHAPTER I

## COLLECTION OF FORMULAS

**1. Formulas for reference.** For the convenience of the student we give the following list of elementary formulas from Algebra, Geometry, Trigonometry, and Analytic Geometry.

1. Binomial Theorem ($n$ being a positive integer):

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-r+2)}{\underline{|r-1}}a^{n-r+1}b^{r-1} + \cdots.$$

Also written:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots$$
$$+ \binom{n}{r-1}a^{n-r+1}b^{r-1} + \cdots.$$

2. $n! = \underline{|n} = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)\,n.$

3. In the quadratic equation $ax^2 + bx + c = 0$,

when $b^2 - 4\,ac > 0$, the roots are real and unequal;
when $b^2 - 4\,ac = 0$, the roots are real and equal;
when $b^2 - 4\,ac < 0$, the roots are imaginary.

4. When a quadratic equation is reduced to the form $x^2 + px = q$,

$p$ = sum of roots with sign changed,
and $q$ = product of roots with sign changed.

5. In an arithmetical series,

$$l = a + (n-1)\,d;\ s = \frac{n}{2}(a+l) = \frac{n}{2}[2\,a + (n-1)\,d$$

6. In a geometrical series,

$$l = ar^{n-1};\ s = \frac{rl - a}{r-1} = \frac{a(r^n - 1)}{r-1}.$$

7. $\log ab = \log a + \log b$.

8. $\log \frac{a}{b} = \log a - \log b$.

9. $\log a^n = n \log a$.

10. $\log \sqrt[n]{a} = \frac{1}{n} \log a$.

11. $\log 1 = 0$.

12. $\log_a a = 1$.

13. $\log \frac{1}{a} = - \log a$.

14. Circumference of circle $= 2\, \pi r$.*

15. Area of circle $= \pi r^2$.

16. Volume of prism $= Ba$.

17. Volume of pyramid $= \frac{1}{3} Ba$.

18. Volume of right circular cylinder $= \pi r^2 a$.

19. Lateral surface of right circular cylinder $= 2\, \pi r a$.

20. Total surface of right circular cylinder $= 2\, \pi r (r + a)$.

21. Volume of right circular cone $= \frac{1}{3} \pi r^2 a$.

22. Lateral surface of right circular cone $= \pi r s$.

23. Total surface of right circular cone $= \pi r (r + s)$.

24. Volume of sphere $= \frac{4}{3} \pi r^3$.

25. Surface of sphere $= 4\, \pi r^2$.

26. $\sin x = \frac{1}{\csc x}$; $\cos x = \frac{1}{\sec x}$; $\tan x = \frac{1}{\cot x}$.

27. $\tan x = \frac{\sin x}{\cos x}$; $\cot x = \frac{\cos x}{\sin x}$.

28. $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $1 + \cot^2 x = \csc^2 x$.

29. $\sin x = \cos\left(\frac{\pi}{2} - x\right)$;

$\cos x = \sin\left(\frac{\pi}{2} - x\right)$;

$\tan x = \cot\left(\frac{\pi}{2} - x\right)$.

30. $\sin(\pi - x) = \sin x$;

$\cos(\pi - x) = - \cos x$;

$\tan(\pi - x) = - \tan x$.

31. $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

32. $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

33. $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

34. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

35. $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$.

36. $\sin 2x = 2 \sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$; $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$.

37. $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$; $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$; $\tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$.

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* In formulas 14–25, $r$ denotes radius, $a$ altitude, $B$ area of base, and $s$ slant height.

**38.** $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

**39.** $1 + \cos x = 2\cos^2\frac{x}{2}$; $1 - \cos x = 2\sin^2\frac{x}{2}$.

**40.** $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$; $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$; $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$.

**41.** $\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

**42.** $\sin x - \sin y = 2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

**43.** $\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

**44.** $\cos x - \cos y = -2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

**45.** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; Law of Sines.

**46.** $a^2 = b^2 + c^2 - 2bc\cos A$; Law of Cosines.

**47.** $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$; distance between points $(x_1, y_1)$ and $(x_2, y_2)$.

**48.** $d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$; distance from line $Ax + By + C = 0$ to $(x_1, y_1)$.

**49.** $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$; coördinates of middle point.

**50.** $x = x_0 + x'$, $y = y_0 + y'$; transforming to new origin $(x_0, y_0)$.

**51.** $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$; transforming to new axes making the angle $\theta$ with old.

**52.** $x = \rho\cos\theta$, $y = \rho\sin\theta$; transforming from rectangular to polar coördinates.

**53.** $\rho = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x}$; transforming from polar to rectangular coördinates.

**54.** Different forms of equation of a straight line:

(a) $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, two-point form;
(b) $\frac{x}{a} + \frac{y}{b} = 1$, intercept form;
(c) $y - y_1 = m(x - x_1)$, slope-point form;
(d) $y = mx + b$, slope-intercept form;
(e) $x\cos\alpha + y\sin\alpha = p$, normal form;
(f) $Ax + By + C = 0$, general form.

**55.** $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$, angle between two lines whose slopes are $m_1$ and $m_2$.

$m_1 = m_2$ when lines are parallel,

and $m_1 = -\frac{1}{m_2}$ when lines are perpendicular.

**56.** $(x - \alpha)^2 + (y - \beta)^2 = r^2$, equation of circle with center $(\alpha, \beta)$ and radius $r$.

## 2. Greek alphabet.

| Letters | Names | Letters | Names |
|---|---|---|---|
| Α α | Alpha | Ν ν | Nu |
| Β β | Beta | Ξ ξ | Xi |
| Γ γ | Gamma | Ο ο | Omicron |
| Δ δ | Delta | Π π | Pi |
| Ε ε | Epsilon | Ρ ρ | Rho |
| Ζ ζ | Zeta | Σ σ ς | Sigma |
| Η η | Eta | Τ τ | Tau |
| Θ θ | Theta | Υ υ | Upsilon |
| Ι ι | Iota | Φ φ | Phi |
| Κ κ | Kappa | Χ χ | Chi |
| Λ λ | Lambda | Ψ ψ | Psi |
| Μ μ | Mu | Ω ω | Omega |

## 3. Natural values of trigonometric functions.

| Angle in Radians | Angle in Degrees | Sin | Cos | Tan | Cot | | |
|---|---|---|---|---|---|---|---|
| .0000 | 0° | .0000 | 1.0000 | .0000 | ∞ | 90° | 1.5708 |
| .0873 | 5° | .0872 | .9962 | .0875 | 11.430 | 85° | 1.4835 |
| .1745 | 10° | .1736 | .9848 | .1763 | 5.671 | 80° | 1.3963 |
| .2618 | 15° | .2588 | .9659 | .2679 | 3.732 | 75° | 1.3090 |
| .3491 | 20° | .3420 | .9397 | .3640 | 2.747 | 70° | 1.2217 |
| .4363 | 25° | .4226 | .9063 | .4663 | 2.145 | 65° | 1.1345 |
| .5236 | 30° | .5000 | .8660 | .5774 | 1.732 | 60° | 1.0472 |
| .6109 | 35° | .5736 | .8192 | .7002 | 1.428 | 55° | .9599 |
| .6981 | 40° | .6428 | .7660 | .8391 | 1.192 | 50° | .8727 |
| .7854 | 45° | .7071 | .7071 | 1.0000 | 1.000 | 45° | .7854 |
| | | Cos | Sin | Cot | Tan | Angle in Degrees | Angle in Radians |

| Angle in Radians | Angle in Degrees | Sin | Cos | Tan | Cot | Sec | Csc |
|---|---|---|---|---|---|---|---|
| $0$ | $0°$ | $0$ | $1$ | $0$ | $\infty$ | $1$ | $\infty$ |
| $\frac{\pi}{2}$ | $90°$ | $1$ | $0$ | $\infty$ | $0$ | $\infty$ | $1$ |
| $\pi$ | $180°$ | $0$ | $-1$ | $0$ | $\infty$ | $-1$ | $\infty$ |
| $\frac{3\pi}{2}$ | $270°$ | $-1$ | $0$ | $\infty$ | $0$ | $\infty$ | $-1$ |
| $2\pi$ | $360°$ | $0$ | $1$ | $0$ | $\infty$ | $1$ | $\infty$ |

| Angle in Radians | Angle in Degrees | Sin | Cos | Tan | Cot | Sec | Csc |
|---|---|---|---|---|---|---|---|
| $0$ | $0°$ | $0$ | $1$ | $0$ | $\infty$ | $1$ | $\infty$ |
| $\frac{\pi}{6}$ | $30°$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ | $\frac{2\sqrt{3}}{3}$ | $2$ |
| $\frac{\pi}{4}$ | $45°$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ | $1$ | $\sqrt{2}$ | $\sqrt{2}$ |
| $\frac{\pi}{3}$ | $60°$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{\sqrt{3}}{3}$ | $2$ | $\frac{2\sqrt{3}}{3}$ |
| $\frac{\pi}{2}$ | $90°$ | $1$ | $0$ | $\infty$ | $0$ | $\infty$ | $1$ |

### 4. Rules for signs.

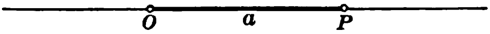
| Quadrant | Sin | Cos | Tan | Cot | Sec | Csc |
|---|---|---|---|---|---|---|
| First . . . . | + | + | + | + | + | + |
| Second. . . . | + | − | − | − | − | + |
| Third . . . . | − | − | + | + | − | − |
| Fourth. . . . | − | + | − | − | + | − |

# CHAPTER II

## NUMBERS

**5. Rational numbers.** All positive and negative integers and fractions, and zero, are called *rational numbers.* We shall assume that the student is familiar with the most elementary properties of these numbers and their use in ordinary Arithmetical work.

**6. Comparison of rational numbers with the points of a straight line.** The series of rational numbers is unlimited, for between any two we can always insert as many more rational numbers as we please. Nevertheless there exist *gaps* everywhere in the series, as may be clearly seen if we set up a correspondence between the series of rational numbers and the points of a straight line.



On a straight line of indefinite length select a zero point $O$ and a definite unit of length for measuring segments. A length may then be constructed corresponding to any rational number $a$ which we lay off to the right or left of $O$ according as $a$ is positive or negative. In this way we obtain a definite end point $P$ which may be considered as the point corresponding to the rational number $a$.* We may then say, *to every rational number there corresponds one and only one point on the straight line.*

But there are lengths which are incommensurable with a given unit of length. From Geometry we have the familiar example of the diagonal of a square whose side is the unit of length. Laying off such a length from the origin on the straight line we obtain an end point which corresponds to no rational number.† And

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* In above figure $a$ is taken as positive.

† Length of diagonal of a unit square $= \sqrt{2}$. This cannot be an integer, for no integer multiplied by itself gives 2. Neither can it be a fraction; for, if possible, let

$$\sqrt{2} = \frac{a}{b}, \qquad \text{(A)}$$

where $a$ and $b$ are integers which do not have a common factor. Squaring both sides,

$$2 = \frac{a^2}{b^2}. \qquad \text{(B)}$$

Since $a$ and $b$ have no common factor, $a^2$ and $b^2$ can have no common factor; and (B), which says that $b^2$ is contained twice in $a^2$, contradicts our hypothesis (A). Therefore, since $\sqrt{2}$ is neither an integer nor a fraction it cannot be a rational number.

since there are infinitely many lengths which are incommensurable with the unit of length, the straight line is infinitely richer in point-individuals than the series of rational numbers is in number-individuals. This comparison has led to the recognition of a certain *incompleteness* of the system of rational numbers, while we ascribe to the straight line *completeness* and *absence of gaps*, that is, *continuity*.

**7. Irrational numbers.** If we wish to study the straight line arithmetically, the system of rational numbers having been found wanting, it becomes necessary to extend our system of numbers in such a way that it shall have the same *completeness*, or *continuity*, as the straight line. This has been done by the creation of *irrational numbers* which are defined in terms of rational numbers only. The scope of this book does not permit the development of the modern arithmetic theory of rational numbers; hence we shall only call the attention of the student to the existence of irrational numbers and to the statement: *the irrational numbers completely fill up all the gaps which exist in the system of rational numbers;* i.e. *we assume that to every point on a straight line corresponds a number, rational or irrational, and conversely.* Following are examples of irrational numbers:

$$\sqrt{2} = 1.4142136\cdots,$$ [*]

$$\log_{10} 5 = 0.6989700\cdots,$$ [†]

$$\pi = 3.1415929\cdots,$$

$$e = 2.7182814\cdots.$$

**8. Real numbers.** All rational and irrational numbers are called *real numbers*. These are arranged in order with respect to their magnitudes as follows:

$$\cdots -4 \cdots -3 \cdots -2 \cdots -1 \cdots 0 \cdots +1 \cdots +2 \cdots +3 \cdots +4 \cdots,$$

increasing as we pass from left to right.

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[*] It was shown in footnote on p. 7 that $\sqrt{2}$ cannot be a rational number.

[†] Suppose this to be a rational number, then

$$\log_{10} 5 = \frac{a}{b},$$

where $a$ and $b$ are positive integers. Then

$$10^{\frac{a}{b}} = 5, \text{ or } 10^a = 5^b.$$

That is, no matter what the values of $a$ and $b$, we would have a number whose last digit is zero equal to a number whose last digit is 5; this being absurd, our hypothesis that $\log_{10} 5$ was a rational number is absurd.

The symbol $>$ is read *is greater than*, and the symbol $<$ is read *is less than.*

It is sometimes convenient to write $a > 0$, which means that $a$ is positive; or $b < 0$, showing that $b$ is negative.

We may also write such expressions as

$$3 > -1, \text{ or } -8 < -5, \text{ etc.}$$

The symbol $\geqq$ is read *is greater than or equal to*, and is equivalent to the symbol $\nless$, read *is not less than.*

The symbol $\leqq$ is read *is less than or equal to*, and is equivalent to the symbol $\ngtr$, read *is not greater than.*

The symbol $\gtrless$ is read *is greater or less than*, and is equivalent to the symbol $\neq$, read *is not equal to.*

**9. Numerical or absolute value.** By the *numerical value* or *absolute value* of a real number we mean its value taken positively. The numerical or absolute value of $a$ is denoted by the symbol $|a|$.

Thus, $$|5| = |-5| = +5.$$

**10. Imaginary numbers.** Consider the equation

$$x^2 + 1 = 0.$$

No real number substituted for $x$ will satisfy this equation. To overcome this difficulty, our number system must be enlarged by the creation of a *new* number. If $i$ is a number such that $i^2 = -1$, then the above equation is satisfied by substituting $i$ or $-i$ for $x$. Hence

$$x = \pm i$$

is called the solution of the equation, and the new number $i = \sqrt{-1}$ is termed the *imaginary unit.*

If $a$ is real, the expression

$$a\sqrt{-1}, \text{ or } ai,$$

defines an *imaginary number.*

**11. Complex numbers.** The sum

$$a + bi,$$

where $a$ and $b$ are real numbers, defines a *complex number.* The first term belongs to the system of real numbers, while the second belongs to the system of imaginary numbers. Complex numbers suffice for all algebraic operations.

**12. Division by zero excluded.** $\frac{0}{0}$ is indeterminate. For, the quotient of two numbers is that number which multiplied by the divisor will give the dividend. But any number whatever multiplied by zero gives zero, and the quotient is indeterminate; that is, any number whatever may be considered as the quotient, a result which is of no value.

$\frac{a}{0}$ has no meaning, $a$ being different from zero, for there exists no number such that if it be multiplied by zero the product would equal $a$.

Therefore *division by zero is not an admissible operation.*

**13. Only real numbers considered.** Unless otherwise stated, only real numbers are considered in what follows in this book.

# CHAPTER III

## VARIABLES AND FUNCTIONS

**14. Variables.** A *variable* is a quantity to which an unlimited number of values can be assigned. Variables are denoted by the later letters of the alphabet. Thus, in the equation of a straight line,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

$x$ and $y$ may be considered as the variable coördinates of a point moving along the line.

**15. Constants.** A quantity whose value remains unchanged is called a *constant.*

*Numerical* or *absolute constants* retain the same values in all problems, as 2, 5, $\sqrt{7}$, $\pi$, etc.

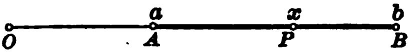
*Arbitrary constants*, or *parameters*, are constants to which any one of an unlimited set of numerical values may be assigned, and they are supposed to have these assigned values throughout the investigation. They are usually denoted by the earlier letters of the alphabet. Thus, for every pair of values arbitrarily assigned to $a$ and $b$, the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

represents some particular straight line.

**16. Interval of a variable.** Very often we confine ourselves to a portion only of the number system. For example, we may restrict our variable so that it shall take on only such values as lie between $a$ and $b$, where $a$ and $b$ may be included, or either or both excluded. We shall employ the symbol $[a, b]$, $a$ being less than $b$, to represent the numbers $a$, $b$, and all the numbers between them, unless otherwise stated. This symbol $[a, b]$ is read *the interval from* $a$ *to* $b$.

**17. Continuous variation.** A variable $x$ is said to vary continuously through an interval $[a, b]$, when $x$ starts with the value $a$ and increases until it takes on the value $b$ in such a manner as to assume the value of every number between $a$ and $b$ in the order of their magnitudes. This may be illustrated geometrically as follows:



The origin being at $O$, lay off on the straight line the points $A$ and $B$ corresponding to the numbers $a$ and $b$. Also let the point $P$ correspond to a particular value of the variable $x$. Evidently the interval $[a, b]$ is represented by the segment $AB$. Now as $x$ varies continuously from $a$ to $b$ inclusive, i.e. through the interval $[a, b]$, the point $P$ generates the segment $AB$.

**18. Functions.** When two variables are so related that the value of the first variable depends on the value of the second variable, then the first variable is said to be a *function* of the second variable.

Nearly all scientific problems deal with quantities and relations of this sort, and in the experiences of everyday life we are continually meeting conditions illustrating the dependence of one quantity on another. For instance, the *weight* a man is able to lift depends on his *strength*, other things being equal. Similarly, the *distance* a boy can run may be considered as depending on the *time*. Or, we may say that the *area* of a square is a function of the *length* of a side, and the *volume* of a sphere is a function of its *diameter*.

**19. Independent and dependent variables.** The second variable, to which values may be assigned at pleasure within limits depending on the particular problem, is called the *independent variable*, or *argument;* and the first variable, whose value is determined as soon as the value of the independent variable is fixed, is called the *dependent variable*, or *function*.

Frequently, when we are considering two related variables, it is in our power to fix upon whichever we please as the *independent variable;* but having once made the choice, no change of independent variable is allowed without certain precautions and transformations.

One quantity (the dependent variable) may be a function of two or more other quantities (the independent variables, or arguments). For example, the *cost* of cloth is a function of both the *quality* and *quantity;* the *area* of a triangle is a function of the *base* and *altitude;* the volume of a rectangular parallelopiped is a function of its *three dimensions.*

**20. Notation of functions.** The symbol $f(x)$ is used to denote a function of $x$, and is read *f function of x.* In order to distinguish between different functions the prefixed letter is changed, as $F(x)$, $\phi(x)$, $f'(x)$, etc.

During any investigation the same functional symbol always indicates the same law of dependence of the function upon the variable. In the simpler cases, this law takes the form of a series of analytical operations upon that variable. Hence, in such a case, the same functional symbol will indicate the same operations or series of operations, even though applied to different quantities. Thus, if

$$f(x)=x^2-9x+14,$$

then

$$f(y)=y^2-9y+14.$$

Also

$$f(a)=a^2-9a+14,$$

$$f(b+1)=(b+1)^2-9(b+1)+14=b^2-7b+6,$$

$$f(0)=0^2-9\cdot 0+14=14,$$

$$f(-1)=(-1)^2-9(-1)+14=24,$$

$$f(3)=3^2-9\cdot 3+14=-4,$$

$$f(7)=7^2-9\cdot 7+14=0, \text{ etc.}$$

Similarly $\phi(x, y)$ denotes a function of $x$ and $y$, and is read *$\phi$ function of x and y.*

If

$$\phi(x, y)=\sin(x+y),$$

then

$$\phi(a, b)=\sin(a+b),$$

and

$$\phi\left(\frac{\pi}{2}, 0\right)=\sin\frac{\pi}{2}=1.$$

Again, if

$$F(x, y, z)=2x+3y-12z,$$

then

$$F(m, -m, m)=2m-3m-12m=-13m,$$

and

$$F(3, 2, 1)=2\cdot 3+3\cdot 2-12\cdot 1=0.$$

Evidently this system of notation may be extended indefinitely.

**21. Values of the independent variable for which a function is defined.** Consider the functions

$$x^2 - 2x + 5, \sin x, \text{arc} \tan x$$

of the independent variable $x$. Denoting the dependent variable in each case by $y$, we may write

$$y = x^2 - 2x + 5, \quad y = \sin x, \quad y = \text{arc} \tan x.$$

In each case $y$ (the value of the function) is known, or, as we say, *defined*, for all values of $x$. This is not by any means true of all functions, as the following examples illustrating the more common exceptions will show.

(1) $y = \frac{a}{x - b}$.

Here the value of $y$ (i.e. the function) is *defined* for all values of $x$ except $x = b$. When $x = b$ the divisor becomes zero and the value of $y$ cannot be computed from (1). Any value might be assigned to the function for this value of the argument.

(2) $y = \sqrt{x}$.

In this case the function is *defined* only for positive values of $x$. Negative values of $x$ give imaginary values for $y$, and these must be excluded here where we are confining ourselves to real numbers only.

(3) $y = \log_a x$.

Here $y$ is *defined* only for positive values of $x$. For negative values of $x$ this function does not exist (see § 35).

(4) $y = \text{arc} \sin x$, $y = \text{arc} \cos x$.

Since sines and cosines cannot become greater than $+1$ nor less than $-1$, it follows that the above functions are *defined* for all values of $x$ ranging from $-1$ to $+1$ inclusive, but for no other values.

**22. One-valued and many-valued functions.** A variable $y$ is said to be a *one-valued function* of a second variable $x$ when $y$ has one and only one value corresponding to each value of $x$. Thus, in

$$y = 3x^2$$

$y$ is a one-valued function of $x$.

If to each value of the second variable there correspond more than one value of the first variable, then the first variable is said to be a *many-valued function* of the second variable. In

$$y^2 = 5x$$

$y$ is a two-valued function of $x$ since

$$y = \pm\sqrt{5x}.$$

Again, in $$y = \text{arc}\tan x$$

it is seen that there is no limit to the number of values of $y$ corresponding to a given value of $x$. For, let $x = 0$, then $y = n\pi$, where $n$ denotes zero or any integer.

**23. Explicit functions.** When a relation between $x$ and $y$ is given by means of an equation *solved for* $y$, then $y$ is called an *explicit function of* $x$. Thus, in

$$y = 2x^3 - x^2 + 3,\ \ y = \frac{2x+1}{\sqrt{x-3}},$$

$$y = \sin ax,\ \ y = \log(1+x),\ \ y = 5a^x,$$

$y$ is in each case an *explicit function of* $x$.

Again, in $$z = \log(x+y)$$

$z$ is an *explicit function of* $x$ *and* $y$.

Similarly $$w = e^{xyz}$$

exhibits $w$ as an *explicit function of* $x$, $y$, *and* $z$.

Symbolically these explicit functions may be respectively denoted by

$$y = f(x),$$
$$z = \phi(x, y),$$
$$w = F(x, y, z).$$

**24. Inverse functions.** Let $y$ be given as a function of $x$ by means of the relation

$$y = f(x).$$

It is usually possible in the case of functions considered in this book to solve this equation for $x$, giving

$$x = \phi(y);$$

that is, to consider $y$ as the independent, and $x$ as the dependent variable. In that case

$$f(x) \text{ and } \phi(y)$$

are said to be *inverse functions*. When we wish to distinguish between the two it is customary to call the first one given the *direct function* and the second one the *inverse function*. Thus, in the examples which follow, if the second members in the first column are taken as the direct functions, then the corresponding members in the second column will be respectively their *inverse functions*.

$$y = x^2 + 1, \qquad x = \pm\sqrt{y-1}.$$
$$y = a^x, \qquad x = \log_a y.$$
$$y = \sin x, \qquad x = \text{arc} \sin y.$$

**25. Integral rational functions.** When $y$ is put equal to an expression which is formed from $x$ and constants by means of addition, subtraction, and multiplication, repeated a finite number of times, then $y$ is said to be an *integral rational function of $x$*, or a *polynomial*. Examples are

$$y = 2x^5 - 5x^2 + 2x - 3, \quad y = ax^2 + bx + c.$$

**26. Rational functions.** When $y$ is equal to an expression which is formed from $x$ and constants by means of the four fundamental operations only, repeated a finite number of times, then $y$ is said to be a *rational function of $x$*. Functions of this sort do not contain radicals and may be reduced to the quotient of two integral rational functions, i.e. to the quotient of two polynomials in $x$ not involving fractional or negative exponents. For example,

$$y = \frac{\dfrac{2x-3}{x-1} + \dfrac{3x-2}{x+1}}{\dfrac{x}{x-2}} + 7x,$$

giving $y$ as a rational function of $x$, may be reduced to the form

$$y = \frac{7x^4 + 5x^3 - 23x^2 + 11x + 2}{x^3 - x}.$$

**27. Explicit algebraic functions.** If $y$ is placed equal to an expression which is formed from $x$ and constants by a finite number of repetitions of the four fundamental operations and the extraction of integral roots, then $y$ is called an *explicit algebraic function of $x$.*[*] Thus, in

$$y = 3x^4 - 2\sqrt{x} - 5,$$
$$y = \frac{2x^3 + x - 6}{3x + 7} + 11x^2 + 9,$$
$$y = \sqrt[3]{\frac{x-1}{x^2+2}},$$

$y$ is in each case an explicit algebraic function of $x$.

Evidently both *rational functions* and *integral rational functions* are *explicit algebraic functions.*

**28. Transcendental functions.** All functions which are *not algebraic* are classed as *transcendental.* The *elementary* transcendental functions are:

(1) *Exponential functions,* in which variables enter as exponents; as $a^x$, $y^x$, $e^{x+y}$, $x^y$, etc.

(2) *Logarithmic functions,* involving the logarithms of the variables; as $\log ax$, $\log(x+y)$, etc.

(3) *Trigonometric functions*[†]; as $\sin ax$, $\cos(x-y)$, $\tan 5z$, etc.

---

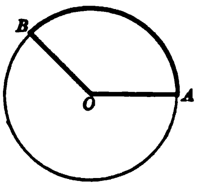
[*] In general $y$ is said to be an *algebraic function of $x$* if $y$ is a root of an equation of the form

$$f_0(x)y^n + f_1(x)y^{n-1} + f_2(x)y^{n-2} + \cdots + f_{n-1}(x)y + f_n(x) = 0,$$

where $f_0(x), f_1(x), f_2(x) \cdots$ are integral rational functions of $x$ (see § 25).

[†] In the further study of mathematics an angle is always understood to be given, not in degrees, minutes, and seconds, but in terms of the radian as unit angle. This unit angle is the angle subtended at the center of a circle by an arc whose length is equal to that of the radius of the circle. The measure of any angle in terms of this unit is the ratio of the length of the arc which subtends the angle to the length of the radius. Thus, in figure,

$$\text{angle } AOB = \frac{\text{arc } AB}{AO}.$$



If arc $AB$ is twice the radius in length, then

$$\text{angle } AOB = 2.$$

Since in any circle the ratio of the circumference to the radius is $2\pi$, we have

$$\text{angle } 2\pi = 360^\circ,$$
$$\text{angle } \pi = 180^\circ,$$
$$\text{angle } 1 = \frac{180^\circ}{\ } = 57^\circ.29+.$$

(4) *Inverse trigonometric functions;* as arc sin $x$,* arc cot$(x - y)$, etc.

Many more transcendental functions are studied in the higher branches of mathematics.

### EXAMPLES

1. Given $f(x) = x^3 - 10x^2 + 31x - 30$; show that

$$\begin{aligned} &f(0) = -30, && f(y) = y^3 - 10y^2 + 31y - 30,\\ &f(2) = 0, && f(a) = a^3 - 10a^2 + 31a - 30,\\ &f(3) = f(5), && f(yz) = y^3z^3 - 10y^2z^2 + 31yz - 30,\\ &f(1) > f(-3), && f(x+2) = x^3 - 16x^2 + 83x - 140,\\ &f(-1) = -6f(6), && f(x+h) = x^3 + (h-10)x^2 - (h^2 - 10h - 31)x\\ &&& \qquad + h^3 - 10h^2 + 31h - 30. \end{aligned}$$

2. If $f(x) = x^3 - 10x^2 + 31x - 30$, and $\phi(x) = x^4 - 55x^2 - 210x - 216$; show that

$$\begin{aligned} f(2) &= \phi(-2),\\ f(3) &= \phi(-3),\\ f(5) &= \phi(-4),\\ f(0) + \phi(0) + 246 &= 0. \end{aligned}$$

3. Given $F(x) = x(x-1)(x+6)(x-\frac{1}{2})(x+\frac{5}{4})$; show that

$$F(0) = F(1) = F(-6) = F(\tfrac{1}{2}) = F(-\tfrac{5}{4}) = 0.$$

4. If $f(m_1) = \dfrac{m_1 - 1}{m_1 + 1}$; show that

$$\frac{f(m_1) - f(m_2)}{1 + f(m_1)f(m_2)} = \frac{m_1 - m_2}{1 + m_1m_2}.$$

5. Given $\phi(x) = \log\dfrac{1-x}{1+x}$; show that

$$\phi(x) + \phi(y) = \phi\left(\frac{x+y}{1+xy}\right).$$

6. If $f(\phi) = \cos\phi$; show that

$$f(\phi) = f(-\phi) = -f(\pi - \phi) = -f(\pi + \phi).$$

7. If $F(\theta) = \tan\theta$; show that

$$F(2\theta) = \frac{2F(\theta)}{1 + [F(\theta)]^2}.$$

8. Given $\psi(x) = x^{2n} + x^{2m} + 1$; show that

$$\psi(1) = 3, \qquad \psi(0) = 1, \qquad \psi(a) = \psi(-a).$$

---

* Also written $\sin^{-1}x$, the $-1$ not being considered as a negative exponent in the ordinary sense, but merely indicating the inverse function. The expression $y = \text{arc} \sin x$ should be read *$y$ equals the arc (or angle) whose sine is $x$*, and the same relation between $x$ and $y$ is given by $\sin y = x$.

For example, since $\tan\frac{\pi}{4} = 1$,

we may also write $\frac{\pi}{4} = \text{arc} \tan 1$.

# CHAPTER IV

## THEORY OF LIMITS

**29. Limit of a variable.** If a variable $v$ takes on successively a series of values that approach nearer and nearer to a constant value $l$ in such a manner that $|v - l|$* becomes and remains less than any assigned arbitrarily small positive quantity, then $v$ is said to *approach the limit* $l$, or to *converge to the limit* $l$. Symbolically this is written

$$\textbf{limit}\ v = l.$$

The following familiar examples illustrate what is meant.

(1) As the number of sides of a regular inscribed polygon is indefinitely increased, the limit of the area of the polygon is the area of the circle. In this case *the variable is always less than its limit.*

(2) Similarly the limit of the area of the circumscribed polygon is also the area of the circle, but now *the variable is always greater than its limit.*

(3) Consider the series

$$(A) \qquad 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots.$$

The sum of any even number $(2n)$ of the first terms of this series is

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{1}{2^{2n-2}} - \frac{1}{2^{2n-1}},$$

$$(B) \qquad S_{2n} = \frac{\frac{1}{2^{2n}} - 1}{-\frac{1}{2} - 1} = \frac{2}{3} - \frac{1}{3 \cdot 2^{2n-1}}. \qquad \text{By 6, p. 1}$$

Similarly the sum of any odd number $(2n + 1)$ of the first terms of the series is

$$S_{2n+1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{2^{2n-1}} + \frac{1}{2^{2n}},$$

$$(C) \qquad S_{2n+1} = \frac{-\frac{1}{2^{2n+1}} - 1}{-\frac{1}{2} - 1} = \frac{2}{3} + \frac{1}{3 \cdot 2^{2n}}. \qquad \text{By 6, p. 1}$$

---

* To be read *the numerical value of the difference between* $v$ *and* $l$.

Writing ($B$) and ($C$) in the forms

$$\frac{2}{3} - S_{2n} = \frac{1}{3 \cdot 2^{2n-1}}; \quad S_{2n+1} - \frac{2}{3} = \frac{1}{3 \cdot 2^{2n}};$$

we have
$$\lim_{n=\infty}\left(\frac{2}{3} - S_{2n}\right) = \lim_{n=\infty}\frac{1}{3 \cdot 2^{2n-1}} = 0,$$

and
$$\lim_{n=\infty}\left(S_{2n+1} - \frac{2}{3}\right) = \lim_{n=\infty}\frac{1}{3 \cdot 2^{2n}} = 0.$$

Hence by definition of the limit of a variable it is seen that both $S_{2n}$ and $S_{2n+1}$ are variables approaching $\frac{2}{3}$ as a limit as the number of terms increases without limit.

Summing up the first two, three, four, etc., terms of ($A$), the sums are found by ($B$) and ($C$) to be alternately less and greater than $\frac{2}{3}$, illustrating the case when *the variable*, in this case the sum of the terms of ($A$), *is sometimes less and sometimes greater than its limit.*

In the examples shown *the variable never reaches its limit.* This is not by any means always the case, for from the definition of the *limit of a variable* it is clear that the essence of the definition is simply that the numerical value of the difference between the variable and its limit shall ultimately become and remain less than any positive number we may choose however small.

(4) As an example illustrating the fact that the variable may reach its limit, consider the following. Let a series of regular polygons be inscribed in a circle, the number of sides increasing indefinitely. Choosing any one of these, construct the circumscribed polygon whose sides touch the circle at the vertices of the inscribed polygon. Let $p_n$ and $P_n$ be the perimeters of the inscribed and circumscribed polygons of $n$ sides and $C$ the circumference of the circle, and suppose the values of a variable $x$ to be as follows:

$$p_n, C, P_n, p_{n+1}, C, P_{n+1}, p_{n+2}, C, P_{n+2}, \text{etc.}$$

Then evidently,

$$\lim_{n=\infty} x = C,$$

and *the limit is reached by the variable.*

**30. Infinitesimals.** A variable $v$ whose limit is zero is called an *infinitesimal.** This is written

$$\text{limit } v = 0,$$

and means that the successive numerical values of $v$ ultimately become and remain less than any positive quantity however small. Such a variable is said to *become indefinitely small* or to *ultimately vanish.*

If $\text{limit } v = l$, then $\text{limit } (v - l) = 0$;

that is, *the difference between a variable and its limit is an infinitesimal.*

Conversely, *if the difference between a variable and a constant is an infinitesimal*, then the variable approaches the constant as a limit.

**31. The concept of infinity** ($\infty$). If a variable $v$ ultimately becomes and remains greater than any assigned positive number however large, we say *$v$ increases without limit*, and write

$$\text{limit } v = +\infty.$$

If a variable $v$ ultimately becomes and remains algebraically less than any assigned negative number, we say *$v$ decreases without limit*, and write

$$\text{limit } v = -\infty.$$

If a variable $v$ ultimately becomes and remains in numerical value greater than any assigned positive number however large, we say *$v$, in numerical value, increases without limit*, or *$v$ becomes infinitely great*,† and write

$$\text{limit } v = \infty.$$

Infinity ($\infty$) is not a number; it simply serves to characterize a particular mode of variation of a variable by virtue of which it increases or decreases without limit.

---

* Hence a constant, no matter how small it may be, is not an infinitesimal.

† On account of the notation used and for the sake of uniformity, the expression limit $v = +\infty$ is sometimes read *$v$ approaches the limit plus infinity.* Similarly limit $v = -\infty$ is read *$v$ approaches the limit minus infinity*, and limit $v = \infty$ is read *$v$, in numerical value, approaches the limit infinity.*

While the above notation is convenient to use in this connection, the student must not forget that infinity is not a limit in the sense in which we defined a limit on page 19, for infinity is not a number at all.

**32. Limiting value of a function.** Given a function $f(x)$.

If the independent variable $x$ takes on any series of values such that

$$\text{limit } x = a,$$

and at the same time the dependent variable $f(x)$ takes on a series of corresponding values such that

$$\text{limit } f(x) = A,$$

then as a single statement this is written

$$\underset{x=a}{\text{limit}} f(x) = A,$$

and is read *the limit of $f(x)$, as $x$ approaches the limit $a$ by any set of values, is $A$.*[*]

**33. Continuous and discontinuous functions.** A function $f(x)$ is said to be *continuous for $x = a$* if the limiting value of the function when $x$ approaches the limit $a$ in any manner is the value assigned to the function for $x = a$. In symbols, if

$$\underset{x=a}{\text{limit}} f(x) = f(a),$$

then $f(x)$ is *continuous* for $x = a$.

The function is said to be *discontinuous* for $x = a$ if this condition is not satisfied. For example, if

$$\underset{x=a}{\text{limit}} f(x) = \infty,$$

the function is discontinuous for $x = a$.

The attention of the student is now called to the following cases which occur frequently.

Case I. As an example illustrating a simple case of a function continuous for a particular value of the variable, consider the function

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

For $x = 1$, $f(x) = f(1) = 3$. Moreover, if $x$ approaches the limit 1 in any manner, the function $f(x)$ approaches 3 as a limit. Hence the function is continuous for $x = 1$.

---

[*] It sometimes happens that $f(x)$ approaches one limit when $x$ approaches $a$, $x$ being always less than $a$; and a different limit when $x$ approaches $a$, $x$ being always greater than $a$. Or, $f(x)$ may approach a limit from one side and not from the other; or it may approach no limit from either side. Evidently the above definition excludes all such exceptional cases.

CASE II. The definition of a continuous function assumes that the function is already defined for $x = a$. If this is not the case, however, it is sometimes possible to assign such a value to the function for $x = a$ that the condition of continuity shall be satisfied. The following theorem covers these cases:

**Theorem.** *If $f(x)$ is not defined for $x = a$, and if*

$$\underset{x=a}{\text{limit}} f(x) = B,$$

*then $f(x)$ will be continuous for $x = a$, if $B$ is assumed as the value of $f(x)$ for $x = a$.* Thus the function

$$\frac{x^2 - 4}{x - 2}$$

is not defined for $x = 2$ (since then there would be division by zero). But for every other value of $x$,

$$\frac{x^2 - 4}{x - 2} = x + 2;$$

and
$$\underset{x=2}{\text{limit}} (x + 2) = 4;$$

therefore
$$\underset{x=2}{\text{limit}} \frac{x^2 - 4}{x - 2} = 4.$$

Although the function is not defined for $x = 2$, if we arbitrarily assign it the value 4 for $x = 2$, it then becomes continuous for this value.

*A function $f(x)$ is said to be continuous in an interval when it is continuous for all values of $x$ in this interval.*[*]

**34. Continuity and discontinuity of functions illustrated by their graphs.**

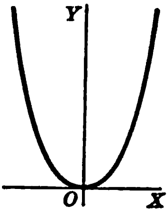
(1) Consider the function $x^2$, and let

$$(A) \qquad y = x^2.$$

---

[*] In this book we shall deal only with functions which are in general continuous, that is, continuous for all values of $x$, with the possible exception of certain isolated values, our results in general being understood as valid only for such values of $x$ for which the function in question is actually continuous. Unless special attention is called thereto, we shall as a rule pay no attention to the possibilities of such exceptional values of $x$ for which the function is discontinuous. The definition of a continuous function $f(x)$ is sometimes roughly (but imperfectly) summed up in the statement that *a small change in $x$ shall produce a small change in $f(x)$*. We shall not consider functions having an infinite number of oscillations in a limited region.

If we assume values for $x$ and calculate the corresponding values of $y$, we can plot a series of points. Drawing a smooth line freehand through these points a good representation of the general behavior of the function may be obtained. This picture or image of the function is called its *graph*. It is evidently the locus of all points satisfying equation ($A$).

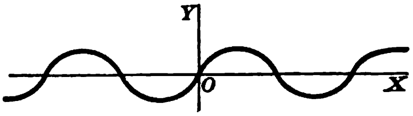


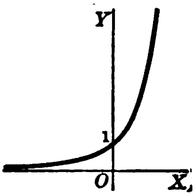
Such a series or assemblage of points is also called a *curve*. Evidently we may assume values of $x$ so near together as to bring the values of $y$ (and therefore the points of the curve) as near together as we please. In other words, there are no breaks in the curve, and the function $x^2$ is continuous for all values of $x$.

(2) The graph of the continuous function $\sin x$ is plotted by drawing the locus of

$$y = \sin x.$$

It is seen that no break in the curve occurs anywhere.





(3) The continuous function $e^x$ is of very frequent occurrence in the Calculus. If we plot its graph from

$$y = e^x, \qquad (e = 2.718 \cdots)$$

we get a smooth curve as shown. From this it is clearly seen that,

(a) when $x = 0$, $\underset{x=0}{\text{limit}}\, y\,( = e^x) = 1$;

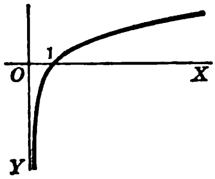
(b) when $x > 0$, $y\,(= e^x)$ is positive and increases as we pass towards the right from the origin;

(c) when $x < 0$, $y\,(= e^x)$ is still positive and decreases as we pass towards the left from the origin.

(4) The function $\log_e x$ is closely related to the last one discussed. In fact, if we plot its graph from

$$y = \log_e x,$$

it will be seen that its graph has the same relation to $OX$ and $OY$ as the graph of $e^x$ has to $OY$ and $OX$.



Here we see the following facts pictured:

(a) For $x = 1$, $\log_e x = \log_e 1 = 0$.

(b) For $x > 1$, $\log_e x$ is positive and increases as $x$ increases.

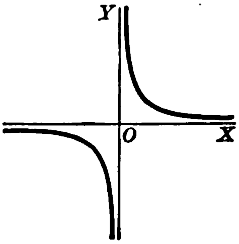
(c) For $1 > x > 0$, $\log_e x$ is negative and increases in *numerical value* as $x$ diminishes.

(d) For $x \leqq 0$, $\log_e x$ is not defined; hence the entire graph lies to the right of $OY$.

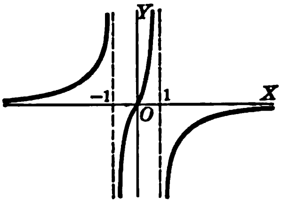
(5) Consider the function $\frac{1}{x}$, and set

$$y = \frac{1}{x}.$$

If the graph of this function be plotted, it will be seen that as $x$ approaches the value zero from the left (negatively) the points of the curve ultimately drop down an infinitely great distance, and as $x$ approaches the value zero from the right the curve extends upward infinitely far.



The curve then does not form a continuous branch from one side to the other of the axis of $Y$, showing graphically that the function is discontinuous for $x = 0$, but continuous for all other values of $x$.



(6) From the graph of

$$y = \frac{2x}{1 - x^2}$$

it is seen that the function

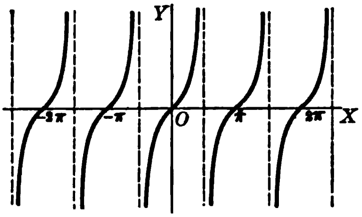
$$\frac{2x}{1 - x^2}$$

is discontinuous for the two values $x = \pm 1$, but continuous for all other values of $x$.

(7) The graph of

$$y = \tan x$$

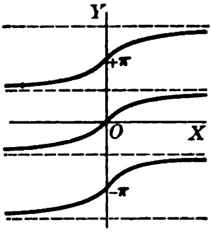
shows that the function $\tan x$ is discontinuous for infinitely many values of $x$, namely, $x = \frac{n\pi}{2}$, where $n$ denotes any odd positive or negative integer.



(8) The function

$$\text{arc}\tan x$$

has infinitely many values for a given value of $x$, the graph of equation



$$y = \text{arc}\tan x$$

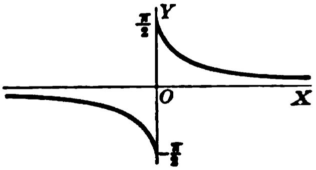
consisting of infinitely many branches. If, however, we confine ourselves to any single branch, the function is continuous. For instance, if we say that $y$ shall be the arc of smallest absolute value whose tangent is $x$, that is, $y$ shall take on only values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, then we are limited to the branch passing through the origin and the condition for continuity is satisfied.

(9) Similarly

$$\text{arc}\tan\frac{1}{x}$$

is found to be a many-valued function. Confining ourselves to one branch of the graph of

$$y = \text{arc}\tan\frac{1}{x},$$



we see that as $x$ approaches zero from the left $y$ approaches the limit $-\frac{\pi}{2}$, and as $x$ approaches zero from the right $y$ approaches the limit $+\frac{\pi}{2}$. Hence the function is discontinuous when $x = 0$. Its value for $x = 0$ can be assigned at pleasure.

Functions exist which are discontinuous for every value of the independent variable within a certain range. In the ordinary applications of the Calculus, however, we deal with functions which are discontinuous (if at all) only for certain isolated values of the independent variable; such functions are therefore in general continuous, and are the only ones considered in this book.

**35. Fundamental theorems on limits.** In problems involving limits the use of one or more of the following theorems is usually implied. It is assumed that the limit of each variable exists and is finite.

**Theorem I.** *The limit of the algebraic sum of a finite number of variables is equal to the like algebraic sum of the limits of the several variables.*

**Theorem II.** *The limit of the product of a finite number of variables is equal to the product of the limits of the several variables.*

**Theorem III.** *The limit of the quotient of two variables is equal to the quotient of the limits of the separate variables, provided the limit of the denominator is not zero.*

Before proving these theorems it is necessary to establish the following properties of infinitesimals.

(1) *The sum of a finite number of infinitesimals is an infinitesimal.* To prove this we must show that the numerical value of this sum can be made less than any small positive quantity (as $\epsilon$) that may be assigned (§ 30). That this is possible is evident, for, the limit of each infinitesimal being zero, each one can be made numerically less than $\frac{\epsilon}{n}$ ($n$ being the number of infinitesimals) and therefore their sum can be made numerically less than $\epsilon$.

(2) *The product of a constant $c$ and an infinitesimal is an infinitesimal.* For the numerical value of the product can always be made less than any small positive quantity (as $\epsilon$) by making the numerical value of the infinitesimal less than $\frac{\epsilon}{c}$.

(3) *The product of any finite number of infinitesimals is an infinitesimal.* For the numerical value of the product may be made less than any small positive quantity that can be assigned. If the given product contains $n$ factors, then since each infinitesimal may be assumed less than the $n$th root of $\epsilon$, the product can be made less than $\epsilon$ itself.

(4) *If $v$ is a variable which approaches a limit $l$ different from zero, then the quotient of an infinitesimal by $v$ is also an infinitesimal.* For if limit $v = l$, and $k$ is any number numerically less than $l$, then by definition of a limit, $v$ will ultimately become and remain numerically greater than $k$. Hence the quotient $\frac{\epsilon}{v}$, where $\epsilon$ is an infinitesimal, will ultimately become and remain numerically less than $\frac{\epsilon}{k}$, and is therefore by (2) an infinitesimal.

*Proof of Theorem I.* Let $v_1, v_2, v_3, \cdots$ be the variables, and $l_1, l_2, l_3, \cdots$ their respective limits. We may then write

$$\begin{aligned} v_1 - l_1 &= \epsilon_1, \\ v_2 - l_2 &= \epsilon_2, \\ v_3 - l_3 &= \epsilon_3, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ are infinitesimals (i.e. variables having zero for a limit). Adding,

$$(A)\quad (v_1 + v_2 + v_3 + \cdots) - (l_1 + l_2 + l_3 + \cdots) = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots).$$

Since the right-hand member is an infinitesimal by (1), p. 27, we have from the converse theorem on p. 21,

$$\text{limit}\,(v_1 + v_2 + v_3 + \cdots) = l_1 + l_2 + l_3 + \cdots$$

or, $\text{limit}\,(v_1 + v_2 + v_3 + \cdots) = \text{limit}\, v_1 + \text{limit}\, v_2 + \text{limit}\, v_3 + \cdots,$

which was to be proved.

*Proof of Theorem II.* Let $v_1$ and $v_2$ be the variables, $l_1$ and $l_2$ their respective limits, and $\epsilon_1$ and $\epsilon_2$ infinitesimals; then

$$v_1 = l_1 + \epsilon_1$$

and $$v_2 = l_2 + \epsilon_2.$$

Multiplying, $$\begin{aligned} v_1 v_2 &= (l_1 + \epsilon_1)(l_2 + \epsilon_2) \\ &= l_1 l_2 + l_1\epsilon_1 + l_2\epsilon_2 + \epsilon_1\epsilon_2, \end{aligned}$$

or,

$$(B)\qquad v_1 v_2 - l_1 l_2 = l_1\epsilon_1 + l_2\epsilon_2 + \epsilon_1\epsilon_2.$$

Since the right-hand member is an infinitesimal by (1) and (2), p. 27, we have as before

$$\text{limit}\,(v_1 v_2) = l_1 l_2 = \text{limit}\, v_1 \cdot \text{limit}\, v_2,$$

which was to be proved.

*Proof of Theorem III.* Using same notation as before,

$$\frac{v_1}{v_2} = \frac{l_1 + \epsilon_1}{l_2 + \epsilon_2} = \frac{l_1}{l_2} + \left(\frac{l_1 + \epsilon_1}{l_2 + \epsilon_2} - \frac{l_1}{l_2}\right),$$

or,

$$(C)\qquad \frac{v_1}{v_2} - \frac{l_1}{l_2} = \frac{l_2\epsilon_1 - l_1\epsilon_2}{l_2(l_2 + \epsilon_2)}.$$

Here again the right-hand member is an infinitesimal by (4), p. 27, if $l_2 \neq 0$, hence

$$\text{limit}\left(\frac{v_1}{v_2}\right)=\frac{l_1}{l_2}=\frac{\text{limit } v_1}{\text{limit } v_2},$$

which was to be proved.

It is evident that if any of the variables be replaced by constants our reasoning still holds and the above theorems are true.

**36. Special limiting values.** The following examples are of special importance in the study of the Calculus. In the first twelve examples $a > 0$ and $c \neq 0$.

| | *Written in the form of limits.* | *Abbreviated form often used.* |
|---|---|---|
| (1) | $\underset{x=0}{\text{limit}}\ \frac{c}{x}=\infty$; | $\frac{c}{0}=\infty$. |
| (2) | $\underset{x=\infty}{\text{limit}}\ cx=\infty$; | $c\cdot\infty=\infty$. |
| (3) | $\underset{x=\infty}{\text{limit}}\ \frac{x}{c}=\infty$; | $\frac{\infty}{c}=\infty$. |
| (4) | $\underset{x=\infty}{\text{limit}}\ \frac{c}{x}=0$; | $\frac{c}{\infty}=0$. |
| (5) | $\underset{x=-\infty}{\text{limit}}\ a^x=+\infty$, when $a<1$; | $a^{-\infty}=+\infty$. |
| (6) | $\underset{x=+\infty}{\text{limit}}\ a^x=0$, when $a<1$; | $a^{+\infty}=0$. |
| (7) | $\underset{x=-\infty}{\text{limit}}\ a^x=0$, when $a>1$; | $a^{-\infty}=0$. |
| (8) | $\underset{x=+\infty}{\text{limit}}\ a^x=+\infty$, when $a>1$; | $a^{+\infty}=+\infty$. |
| (9) | $\underset{x=0}{\text{limit}}\ \log_a x=+\infty$, when $a<1$; | $\log_a 0=+\infty$. |
| (10) | $\underset{x=+\infty}{\text{limit}}\ \log_a x=-\infty$, when $a<1$; | $\log_a(+\infty)=-\infty$. |
| (11) | $\underset{x=0}{\text{limit}}\ \log_a x=-\infty$, when $a>1$; | $\log_a 0=-\infty$. |
| (12) | $\underset{x=+\infty}{\text{limit}}\ \log_a x=+\infty$, when $a>1$; | $\log_a +\infty=+\infty$. |

The expressions in the second column are not to be considered as expressing numerical equalities ($\infty$ not being a number); they are merely *symbolical equations* implying the relations indicated in the first column, and should be so understood.

(13) Find $\underset{x=a}{\text{limit}}\ \dfrac{x^n - a^n}{x - a}$, where $n$ denotes any positive **integer.**[*]

By division we get

$$\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}$$

for every value of $x$ except $x = a$. Therefore

$$\underset{x=a}{\text{limit}}\ \frac{x^n - a^n}{x - a} = \underset{x=a}{\text{limit}}\ x^{n-1} + \underset{x=a}{\text{limit}}\ ax^{n-2} + \cdots + \underset{x=a}{\text{limit}}\ a^{n-2}x + \underset{x=a}{\text{limit}}\ a^{n-1}.$$

[By Theorem I, p. 27.]

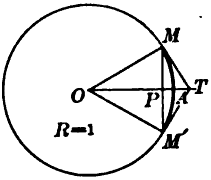
The limit of each term in the second member is $a^n$; and since there are $n$ terms, we have

$$\underset{x=a}{\text{limit}}\ \frac{x^n - a^n}{x - a} = na^{n-1}.$$

(14) Show that $\underset{x=0}{\text{limit}}\ \dfrac{\sin x}{x} = 1.$

Let $O$ be the center of a circle whose radius is unity.

Let arc $AM =$ arc $AM' = x$, and let $MT$ and $M'T$ be tangents drawn to the circle at $M$ and $M'$. From Geometry,



$$MPM' < MAM' < MTM';$$

or, $$2\sin x < 2x < 2\tan x.$$

Dividing through by $2\sin x$ we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

If now $x$ approaches the limit zero,

$$\underset{x=0}{\text{limit}}\ \frac{x}{\sin x}$$

must lie between the constant 1 and $\underset{x=0}{\text{limit}}\ \dfrac{1}{\cos x}$, which is also 1.

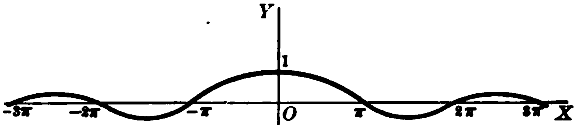
Therefore $\underset{x=0}{\text{limit}}\ \dfrac{x}{\sin x} = 1$, or $\underset{x=0}{\text{limit}}\ \dfrac{\sin x}{x} = 1.$ Th. III, p. 27

---

[*] Restricted to a positive integer in order to simplify the work. The *result* holds true for all values of $n$.

It is interesting to note the behavior of this function from its graph, the locus of equation

$$y = \frac{\sin x}{x}.$$



Although the function is not defined for $x = 0$, yet it is not discontinuous when $x = 0$ if we define

$$\frac{\sin 0}{0} = 1. \qquad \text{Case II, p. 23}$$

(15) Find the limit of the sum of the series

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{n-1}},$$

as the number of terms increases without limit.

By formula 6, p. 1, we find that the sum of $n$ terms of the series is

$$S_n = \frac{(\frac{1}{3})^n - 1}{\frac{1}{3} - 1} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right).$$

Hence

$$\operatorname*{limit}_{n=\infty} S_n = \operatorname*{limit}_{n=\infty} \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

$$= \frac{3}{2} \operatorname*{limit}_{n=\infty}\left(1 - \frac{1}{3^n}\right)$$

$$= \frac{3}{2}\left[\operatorname*{limit}_{n=\infty}(1) - \operatorname*{limit}_{n=\infty}\left(\frac{1}{3^n}\right)\right]$$

$$= \frac{3}{2}\left[1 - 0\right] = \frac{3}{2}. \quad \textit{Ans.}$$

**37. The number e.*** We first proceed to prove two important theorems.

**Theorem I.** *If $a$ is a variable $> -1$ which varies continuously in any interval not including zero, then the function*

$$\phi(a) = (1 + a)^{\frac{1}{a}}$$

*varies in a sense contrary to $a$.*

---

* The proofs in this section are due to Vallée-Poussin.

We start with the identity

$$a^{n+1} - 1 = (a-1)(a^n + a^{n-1} + \cdots + a + 1),$$

or,
$$a^{n+1} = 1 + (a-1)(a^n + a^{n-1} + \cdots + a + 1);$$

where $a$ denotes any positive number and $n$ a positive integer.

Since the last parenthesis is $>$ or $< (n+1)$ according as $a$ is $>$ or $< 1$, or according as $(a-1)$ is positive or negative, we have in either case

$$a^{n+1} > 1 + (n+1)(a-1).$$

Let $\omega$ be any number $> -n$ and different from zero, and replace $a$ in this inequality by the quotient

$$\left(1 + \frac{\omega}{n+1}\right) : \left(1 + \frac{\omega}{n}\right).$$

Multiplying both members by $\left(1 + \frac{\omega}{n}\right)^{n+1}$ and reducing gives

$$\left(1 + \frac{\omega}{n+1}\right)^{n+1} > \left(1 + \frac{\omega}{n}\right)^{n}.$$

Let $m$ be any integer $> n$; then by repeated applications of the last result we see that

$$\left(1 + \frac{\omega}{m}\right)^{m} > \left(1 + \frac{\omega}{n}\right)^{n};$$

and it readily* follows that

$$\left(1 + \frac{\omega}{m}\right)^{\frac{m}{\omega}} \gtrless \left(1 + \frac{\omega}{n}\right)^{\frac{n}{\omega}},$$

according as $\omega \gtrless 0$. Now replacing $\omega$ by $ma$ we get

$$(A) \qquad (1 + a)^{\frac{1}{a}} \gtrless \left(1 + \frac{m}{n} a\right)^{\frac{n}{ma}},$$

according as $a \gtrless 0$. This proves our statement for two values (as $a$ and $\frac{m}{n} a$) having the same sign and whose quotient is rational. In order to extend this proof to the case of two values $a$ and $\beta$ whose ratio is irrational, consider first the case when $a > 0$.

---

[*] The assumption is here made that raising both members of an inequality to any power, *rational or irrational*, does not or does change the sense of the inequality according as the power is positive or negative.

Assuming $a < \beta$ we may so choose $m$ and $n$ that

$$a < \frac{m}{n} a < \beta.$$

Now let $\frac{m}{n} a$ approach $\beta$ as a limit by a series of increasing values; then the second member of the inequality $(A)$ will be constantly decreasing and we get in the limit since the function is continuous

$$(1 + a)^{\frac{1}{a}} > (1 + \beta)^{\frac{1}{\beta}},$$

or,

$$\phi(a) > \phi(\beta). \qquad a < \beta$$

Similarly when $a < 0$ we shall get

$$\phi(a) < \phi(\beta), \qquad a > \beta$$

which establishes the theorem for all cases.

**Theorem II.** *Definition of the number e. As a approaches the limit zero, the function*

$$\phi(a) = (1 + a)^{\frac{1}{a}}$$

*of Theorem I approaches a limit. In whatever manner a approaches zero, the limit is the same number. This limit is denoted by e.*

Consider the two variables $a$ and $\beta$ connected by the condition

$$1 + a = \frac{1}{1 + \beta}, \text{ where } 0 > a > -1;$$

then it follows that

$$\phi(a) = (1 + \beta)\phi(\beta).^{*}$$

If $a$ tends towards zero through a series of increasing (therefore negative) values, $\beta$ will be positive and tend towards zero through a series of decreasing values. Then, by Theorem I, we know that $\phi(a)$ continually decreases and $\phi(\beta)$ continually increases. But $\phi(a)$ always remains positive and therefore must tend towards some definite limit (see Theorem II, above). Denoting this limit by $e$, the last equation, in which $1 + \beta$ tends towards the limit unity, becomes in the limit

---

* $\phi(a) = (1 + a)^{\frac{1}{a}} = \left(1 + \frac{1}{1+\beta} - 1\right)^{\frac{1}{\frac{1}{1+\beta} - 1}}$ $\left[a = \frac{1}{1+\beta} - 1 \text{ by hypothesis.}\right]$

$= \left(\frac{1}{1+\beta}\right)^{-\frac{1+\beta}{\beta}} = (1+\beta)^{\frac{1+\beta}{\beta}} = (1+\beta)(1+\beta)^{\frac{1}{\beta}} = (1+\beta)\phi(\beta).$

$$e = \underset{a=0}{\text{limit}}\,\phi(a) = \underset{\beta=0}{\text{limit}}\,\phi(\beta).$$

Since $a$ was negative and $\beta$ positive, this proves that the limit is the same whatever may be the sign of the variable.

To evaluate this limit we note that $\phi(\beta)$ increases towards its limit, while $\phi(a) = (1 + \beta)\phi(\beta)$ decreases towards the same limit ($= e$). Hence for all positive values of $\beta$,

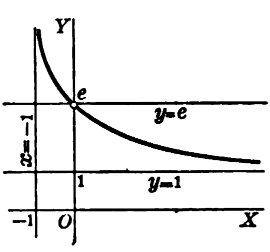
$$\phi(\beta) < e < (1 + \beta)\phi(\beta).$$

By means of this inequality we may calculate the value of $e$ to any desired degree of accuracy by choosing $\beta$ sufficiently small. If we let $\beta = \frac{1}{2}$, then

$$(\tfrac{3}{2})^2 < e < (\tfrac{3}{2})^3;$$

hence $e$ certainly lies between 2 and 4. In Chapter XX, Ex. 14, p. 237, we give a more expeditious method for calculating $e$. Approximately

$$e = 2.71828\cdots.$$



Plotting the graph of $\phi(a)$ from

$$y = (1 + a)^{\frac{1}{a}},$$

and assigning to $y$ the value $e$ when $a = 0$, we see that as $a$ increases without limit $y$ approaches the limit 1, and as $x$ approaches the limit $-1$ from the right $y$ increases without limit.

*Natural logarithms* are those which have the number $e$ for base. These logarithms play a very important rôle in mathematics. When the base is not indicated explicitly, the base $e$ is always understood in what follows in this book. Thus $\log_e v$ is written simply $\log v$.

Natural logarithms possess the following characteristic property: If $a$ approaches zero as a limit in any way whatever,

$$\text{limit}\,\frac{\log(1 + a)}{a} = \text{limit}\,\log(1 + a)^{\frac{1}{a}} = \log e = 1.$$

## EXAMPLES

**Prove the following.**

1. $\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{x+1}{x}\right)=1.$

*Proof.*

$$\begin{aligned}\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{x+1}{x}\right)&=\mathop{\text{limit}}\limits_{x=\infty}\left(1+\frac{1}{x}\right)\\&=\mathop{\text{limit}}\limits_{x=\infty}(1)+\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{1}{x}\right)\qquad\text{Th. I, p. 27}\\&\qquad\text{[Since these limits exist.]}\\&=1+0=1.\end{aligned}$$

2. $\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{x^2+2x}{5-3x^2}\right)=-\frac{1}{3}.$

*Proof.*

$$\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{x^2+2x}{5-3x^2}\right)=\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{1+\frac{2}{x}}{\frac{5}{x^2}-3}\right)$$

[Dividing both numerator and denominator by $x^2$.]

$$=\frac{\mathop{\text{limit}}\limits_{x=\infty}\left(1+\frac{2}{x}\right)}{\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{5}{x^2}-3\right)}\qquad\text{Th. III, p. 27}$$

[Since the limit of the denominator is not zero.]

$$=\frac{\mathop{\text{limit}}\limits_{x=\infty}(1)+\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{2}{x}\right)}{\mathop{\text{limit}}\limits_{x=\infty}\left(\frac{5}{x^2}\right)-\mathop{\text{limit}}\limits_{x=\infty}(3)}\qquad\text{Th. I, p. 27}$$

[Since these limits exist.]

$$=\frac{1+0}{0-3}=-\frac{1}{3}.$$

3. $\mathop{\text{limit}}\limits_{x=1}\frac{x^2-2x+5}{x^2+7}=\frac{1}{2}.$

4. $\mathop{\text{limit}}\limits_{x=0}\frac{3x^3+6x^2}{2x^4-15x^2}=-\frac{2}{5}.$

5. $\mathop{\text{limit}}\limits_{x=-2}\frac{x^2+1}{x+3}=5.$

6. $\mathop{\text{limit}}\limits_{h=0}(3ax^2-2hx+5h^2)=3ax^2.$

7. $\mathop{\text{limit}}\limits_{x=\infty}(ax^2+bx+c)=\infty.$

8. $\mathop{\text{limit}}\limits_{k=0}\frac{(x-k)^2-2kx^3}{x(x+k)}=1.$

9. $\mathop{\text{limit}}\limits_{m=0}[2\sin(a+mx)\cos(a-mx)]=\sin 2a.$

10. $\mathop{\text{limit}}\limits_{u=v}\left[\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)\right]=\sin v.$

11. $\mathop{\text{limit}}\limits_{y=\pi}[\tan^2(y-\theta)-\cos y]=\sec^2\theta.$

12. $\underset{\alpha = \frac{\pi}{2}}{\text{limit}} \dfrac{\cos(\alpha - a)}{\cos(2\alpha - a)} = -\tan a.$

13. $\underset{x = 0}{\text{limit}} \left(\text{arc}\tan \dfrac{1 + x}{x}\right) = \dfrac{n\pi}{2}$, $n$ being an odd integer.

14. $\underset{y = 1}{\text{limit}} (\text{arc}\cos \sqrt{1 - y^2}) = \dfrac{n\pi}{2}$, $n$ being an odd integer.

15. $\underset{x = 0}{\text{limit}} \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}) = a.$

16. $\underset{x = 0}{\text{limit}} \dfrac{2x^3 + 3x^2}{x^3} = \infty.$

17. $\underset{x = \infty}{\text{limit}} \dfrac{5x^2 - 2x}{x} = \infty.$

18. $\underset{y = \infty}{\text{limit}} \dfrac{y}{y + 1} = 1.$

19. $\underset{n = \infty}{\text{limit}} \dfrac{n(n + 1)}{(n + 2)(n + 3)} = 1.$

20. $\underset{s = 1}{\text{limit}} \dfrac{s^3 - 1}{s - 1} = 3.$

21. $\underset{h = 0}{\text{limit}} \dfrac{(x + h)^n - x^n}{h} = nx^{n-1}.$

22. $\underset{h = 0}{\text{limit}} \left[\cos(\theta + h)\dfrac{\sin h}{h}\right] = \cos\theta.$

23. $\underset{\phi = 0}{\text{limit}} \dfrac{\tan\phi}{\phi} = 1.$

24. $\underset{\theta = 0}{\text{limit}} \dfrac{1 - \cos\theta}{\theta^2} = \dfrac{1}{2}.$

25. $\underset{x = a}{\text{limit}} \dfrac{1}{x - a} = -\infty$, if $x$ is increasing as it approaches the value $a$.

26. $\underset{x = a}{\text{limit}} \dfrac{1}{x - a} = +\infty$, if $x$ is decreasing as it approaches the value $a$.

# CHAPTER V

## DIFFERENTIATION

**38. Introduction.** We shall now proceed to investigate the manner in which a function changes in value as the independent variable changes. The fundamental problem of the Differential Calculus is to establish a measure of this change in the function with mathematical precision. It was while investigating problems of this sort, dealing with continuously varying quantities, that Newton* was led to the discovery of the fundamental principles of the Calculus, the most scientific and powerful tool of the modern mathematician.

**39. Increments.** The *increment* of a variable in changing from one numerical value to another is the *difference* found by subtracting the first value from the second. An increment of $x$ is denoted by the symbol $\Delta x$, read *delta x.*

The student is warned against reading this symbol "delta times $x$," it having no such meaning. Evidently this increment may be either positive or negative† according as the variable in changing is increasing or decreasing in value. Similarly

$\Delta y$ denotes an increment of $y$,
$\Delta \phi$ denotes an increment of $\phi$,
$\Delta f(x)$ denotes an increment of $f(x)$, etc.

If in $y = f(x)$ the independent variable $x$ takes on an increment $\Delta x$, then $\Delta y$ is always understood to denote the corresponding increment of the function $f(x)$ (or dependent variable $y$).

---

* Sir Isaac Newton (1642–1727), an Englishman, was a man of the most extraordinary genius. He developed the science of the Calculus under the name of Fluxions. Although Newton had discovered and made use of the new science as early as 1670, his first published work in which it occurs is dated 1687, having the title *Philosophiae Naturalis Principia Mathematica.* This was Newton's principal work. Laplace said of it, "It will always remain preëminent above all other productions of the human mind."

† Some writers call a *negative increment* a *decrement.*

The increment $\Delta y$ is always assumed to be reckoned from a definite initial value of $y$ corresponding to the arbitrarily fixed initial value of $x$ from which the increment $\Delta x$ is reckoned. For instance, consider the function

$$y = x^2.$$

Assuming $x = 10$ for the initial value of $x$ fixes $y = 100$ as the initial value of $y$.

Suppose $x$ increases to $x = 12$, that is, $\Delta x = 2$;
then $y$ increases to $y = 144$, and $\Delta y = 44$.
Suppose $x$ decreases to $x = 9$, that is, $\Delta x = -1$;
then $y$ decreases to $y = 81$, and $\Delta y = -19$.

It may happen that as $x$ increases $y$ decreases, or the reverse; in either case $\Delta x$ and $\Delta y$ will have opposite signs.

It is also clear (as illustrated in the above example) that if

$$y = f(x)$$

is a continuous function and $\Delta x$ is decreasing in numerical value, then $\Delta y$ also decreases in numerical value.

**40. Comparison of increments.** Consider the function

$$(A) \qquad y = x^2.$$

Assuming a fixed initial value for $x$, let $x$ take on an increment $\Delta x$. Then $y$ will take on a corresponding increment $\Delta y$, and we have

$$y + \Delta y = (x + \Delta x)^2,$$

or,

$$y + \Delta y = x^2 + 2\,x \cdot \Delta x + (\Delta x)^2.$$

Subtracting $(A)$,

$$\begin{array}{lrl} & y & = x^2 \\ \hline (B) & \Delta y & = 2\,x \cdot \Delta x + (\Delta x)^2 \end{array}$$

we get the increment $\Delta y$ in terms of $x$ and $\Delta x$.

To find the ratio of the increments, divide $(B)$ by $\Delta x$, giving

$$\frac{\Delta y}{\Delta x} = 2\,x + \Delta x.$$

If the initial value of $x$ is 4, it is evident that

$$\underset{\Delta x = 0}{\text{limit}} \frac{\Delta y}{\Delta x} = 8.$$

Let us carefully note the behavior of the ratio of the increments of $x$ and $y$ as the increment of $x$ diminishes.

| Initial value of $x$ | New value of $x$ | Increment $\Delta x$ | Initial value of $y$ | New value of $y$ | Increment $\Delta y$ | $\frac{\Delta y}{\Delta x}$ |
|---|---|---|---|---|---|---|
| 4 | 5.0 | 1.0 | 16 | 25. | 9. | 9. |
| 4 | 4.8 | 0.8 | 16 | 23.04 | 7.04 | 8.8 |
| 4 | 4.6 | 0.6 | 16 | 21.16 | 5.16 | 8.6 |
| 4 | 4.4 | 0.4 | 16 | 19.36 | 3.36 | 8.4 |
| 4 | 4.2 | 0.2 | 16 | 17.64 | 1.64 | 8.2 |
| 4 | 4.1 | 0.1 | 16 | 16.81 | 0.81 | 8.1 |
| 4 | 4.01 | 0.01 | 16 | 16.0801 | 0.0801 | 8.01 |

It is apparent that as $\Delta x$ decreases $\Delta y$ also diminishes, but their ratio takes on the successive values 9, 8.8, 8.6, 8.4, 8.2, 8.1, 8.01; illustrating the fact that $\frac{\Delta y}{\Delta x}$ can be brought as near to 8 in value as we please by making $\Delta x$ small enough. Therefore

$$\underset{\Delta x = 0}{\text{limit.}} \frac{\Delta y}{\Delta x} = 8.$$

**41. Derivative of a function of one variable.** The fundamental definition of the Differential Calculus is:

*The derivative*[*] *of a function is the limit of the ratio of the increment of the function to the increment of the independent variable, when the latter increment approaches the limit zero.*

When the limit of this ratio exists, the function is said to be *differentiable*, or to *possess a derivative*.

The above definition may be given in a more compact form symbolically as follows: Given

$$(A) \qquad y = f(x),$$

and assume for $x$ some value for which $f(x)$ is continuous.

Let $x$ take on an increment $\Delta x$; then $y$ takes on an increment $\Delta y$, the new value of the function being

$$(B) \qquad y + \Delta y = f(x + \Delta x).$$

---

[*] Also called the *differential coefficient* or the *derived function*.

To find the increment of function, subtract (*A*) from (*B*), giving

$$(C) \qquad \Delta y = f(x+\Delta x) - f(x).$$

Dividing by the increment of the variable $\Delta x$, we get

$$(D) \qquad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The limit of this ratio when $\Delta x$ approaches the limit zero is, from our definition, the *derivative* and is *denoted by the symbol* $\frac{dy}{dx}$. Therefore

$$(E) \qquad \frac{dy}{dx} = \underset{\Delta x = 0}{\text{limit}} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

defines the *derivative of $y$ [or $f(x)$] with respect to $x$.*

From (*D*) we also get

$$\frac{dy}{dx} = \underset{\Delta x = 0}{\text{limit}} \frac{\Delta y}{\Delta x}.$$

The process of finding the derivative of a function is called *differentiation.*

It should be carefully noted that the derivative is the *limit of the ratio*, not the ratio of the limits. The latter ratio would assume the form $\frac{0}{0}$, which is indeterminate (§ 12, p. 10).

**42. Symbols for derivatives.** Since $\Delta y$ and $\Delta x$ are always finite and have definite values, the expression

$$\frac{\Delta y}{\Delta x}$$

is really a fraction. The symbol

$$\frac{dy}{dx},$$

however, is to be regarded, *not as a fraction, but as the limiting value of a fraction.* In many cases it will be seen that this symbol does possess fractional properties, and later on we shall show how meanings may be attached to $dy$ and $dx$, but for the present the symbol $\frac{dy}{dx}$ is to be considered as a whole.

Since the derivative of a function of $x$ is in general also a function of $x$, the symbol $f'(x)$ is also used to denote the derivative of $f(x)$. Hence, if

$$y = f(x),$$

we may write
$$\frac{dy}{dx} = f'(x),$$

which is read *the derivative of $y$ with respect to $x$ equals $f$ prime of $x$*. The symbol

$$\frac{d}{dx}$$

when considered by itself is called the *differentiating operator*, and indicates that any function written after it is to be differentiated with respect to $x$. Thus

$\frac{dy}{dx}$ or $\frac{d}{dx}y$ indicates the derivative of $y$ with respect to $x$;

$\frac{d}{dx}f(x)$ indicates the derivative of $f(x)$ with respect to $x$;

$\frac{d}{dx}(2x^2 + 5)$ indicates the derivative of $2x^2 + 5$ with respect to $x$.

The symbol $D_x$ is used by some writers instead of $\frac{d}{dx}$. If then

$$y = f(x),$$

we may write the identities

$$\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x) = D_x f(x) = f'(x).$$

**43. Differentiable functions.** From the Theory of Limits it is clear that if the derivative of a function exists for a certain value of the independent variable, the function itself must be continuous for that value of the variable.

The converse, however, is not always true; functions having been discovered that are continuous and yet possess no derivative. But such functions do not occur often in applied mathematics, and *in this book only differentiable functions are considered*, that is, functions that possess a derivative for all values of the independent variable save at most for isolated values.

**44. General rule for differentiation.** From the definition of a derivative it is seen that the process of differentiating a function $y=f(x)$ consists in taking the following distinct steps:

### GENERAL RULE FOR DIFFERENTIATION

**First step.** *In the function replace* $x$ *by* $x+\Delta x$, *giving a new value of the function,* $y+\Delta y$.

**Second step.** *Subtract the given value of the function from the new value in order to find* $\Delta y$ *(the increment of the function).*

**Third step.** *Divide the remainder* $\Delta y$ *(the increment of the function) by* $\Delta x$ *(the increment of the independent variable).*

**Fourth step.** *Find the limit of this quotient, when* $\Delta x$ *(the increment of the independent variable) approaches the limit zero. This is the derivative required.*

The student should become thoroughly familiar with this rule by applying the process to a large number of examples. Three such examples will now be worked out in detail.

Ex. 1. Differentiate $3x^2+5$.

*Solution.* Applying the successive steps in the *General Rule* we get, after placing

$$y=3x^2+5,$$

*First step.*

$$\begin{aligned} y+\Delta y &= 3(x+\Delta x)^2+5 \\ &= 3x^2+6x\cdot\Delta x+3(\Delta x)^2+5. \end{aligned}$$

*Second step.*

$$\begin{array}{rll} y+\Delta y &= 3x^2+6x\cdot\Delta x+3(\Delta x)^2 & +5 \\ y &= 3x^2 & +5 \\ \hline \Delta y &= \qquad 6x\cdot\Delta x+3(\Delta x)^2. & \end{array}$$

*Third step.*

$$\frac{\Delta y}{\Delta x}=6x+3\cdot\Delta x.$$

*Fourth step.*

$$\frac{dy}{dx}=6x. \quad \textit{Ans.}$$

We may also write this

$$\frac{d}{dx}(3x^2+5)=6x.$$

Ex. 2. Differentiate $x^3-2x+7$.

*Solution.* Place $y=x^3-2x+7$.

*First step.*

$$\begin{aligned} y+\Delta y &= (x+\Delta x)^3-2(x+\Delta x)+7 \\ &= x^3+3x^2\cdot\Delta x+3x\cdot(\Delta x)^2+(\Delta x)^3-2x-2\cdot\Delta x+7. \end{aligned}$$

*Second step.*

$$\begin{array}{rll} y + \Delta y &= x^3 + 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 - 2x - 2 \cdot \Delta x + 7 \\ y &= x^3 \qquad\qquad\qquad\qquad\qquad\qquad\quad - 2x \qquad\quad + 7 \\ \hline \Delta y &= \qquad 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 \qquad - 2 \cdot \Delta x. \end{array}$$

*Third step.* $$\frac{\Delta y}{\Delta x} = 3x^2 + 3x \cdot \Delta x + (\Delta x)^2 - 2.$$

*Fourth step.* $$\frac{dy}{dx} = 3x^2 - 2. \quad \textit{Ans.}$$

Or, $$\frac{d}{dx}(x^3 - 2x + 7) = 3x^2 - 2.$$

Ex. 3. Differentiate $\frac{c}{x^2}$.

*Solution.* Place $$y = \frac{c}{x^2}.$$

*First step.* $$y + \Delta y = \frac{c}{(x + \Delta x)^2}.$$

*Second step.* $$\Delta y = \frac{c}{(x + \Delta x)^2} - \frac{c}{x^2} = \frac{-c \cdot \Delta x(2x + \Delta x)}{x^2(x + \Delta x)^2}.$$

*Third step.* $$\frac{\Delta y}{\Delta x} = -c \cdot \frac{2x + \Delta x}{x^2(x + \Delta x)^2}.$$

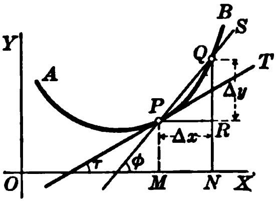
*Fourth step.* $$\frac{dy}{dx} = -c \cdot \frac{2x}{x^2(x)} = -\frac{2c}{x^3}. \quad \textit{Ans.}$$

Or, $$\frac{d}{dx}\left(\frac{c}{x^2}\right) = \frac{-2c}{x^3}.$$

**45. Applications of the derivative to Geometry.** We shall now consider a theorem which is fundamental in all applications of the Differential Calculus to Geometry. Let

$$y = f(x)$$

be the equation of a curve $AB$. Consider a fixed point $P$ whose coördinates are $(x, y)$. Let $x$ take on an increment $\Delta x (= MN)$; then $y$ takes on an increment $\Delta y (= RQ)$, the coördinates of $Q$ being $(x + \Delta x, y + \Delta y)$.



From the figure, $MP = y = f(x)$

and $\qquad NQ = y + \Delta y = f(x + \Delta x)$;

therefore $\quad RQ = \Delta y = f(x + \Delta x) - f(x)$.

Draw a secant line through $P$ and $Q$ and a tangent line to the curve at $P$. Then

$$\tan \phi = \tan QPR = \frac{RQ}{PR} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \text{slope of secant line } PQS.$$

If we now let $\Delta x$ approach the limit zero, the point $Q$ will move along the curve and approach nearer and nearer to $P$, the secant will turn about $P$ and approach the tangent as a limiting position, and we may write

$$\begin{aligned}\tan\tau &= \underset{\Delta x = 0}{\text{limit}}\,(\tan\phi)\\ &= \underset{\Delta x = 0}{\text{limit}}\,\frac{\Delta y}{\Delta x}\\ &= \underset{\Delta x = 0}{\text{limit}}\,\frac{f(x+\Delta x)-f(x)}{\Delta x}, \text{ or,}\\ \tan\tau &= \frac{dy}{dx} \qquad \text{from } (E), \text{ p. } 40\\ &= \text{slope of tangent line } PT. \quad \text{Hence}\end{aligned}$$

**Theorem.** *The value of the derivative at any point of a curve is equal to the slope of the line drawn tangent to the curve at that point.*

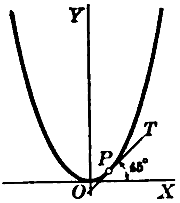
It was this tangent problem that led Leibnitz* to the discovery of the Differential Calculus.

Ex. 1. Find the slopes of the tangents to the parabola $y = x^2$ at the vertex, and at the point where $x = \frac{1}{2}$.

*Solution.* Differentiating by *General Rule*, p. 42, we get

$$(A) \qquad \frac{dy}{dx} = 2x = \text{slope of tangent line at } any \text{ point on curve.}$$

To find slope of tangent at vertex, substitute $x = 0$ in $(A)$,



giving $$\frac{dy}{dx} = 0.$$

Therefore the tangent at vertex has the slope zero, that is, it is parallel to the axis of $x$ and in this case coincides with it.

To find slope of tangent at the point $P$, where $x = \frac{1}{2}$, substitute in $(A)$, giving

$$\frac{dy}{dx} = 1,$$

that is, the tangent at the point $P$ makes an angle of 45° with the axis of $x$.

---

* Gottfreid Wilhelm Leibnitz (1646-1716) was a native of Leipzig. His remarkable abilities were shown by original investigations in several branches of learning. He was first to publish his discoveries in Calculus in a short essay appearing in the periodical *Acta Eruditorum* at Leipzig in 1684. It is known, however, that manuscripts on Fluxions written by Newton were already in existence, and from these some claim Leibnitz got the new ideas. The decision of modern times seems to be that both Newton and Leibnitz invented the Calculus independently of each other. The notation used to-day was introduced by Leibnitz.

## EXAMPLES

Use the *General Rule*, p. 42, in differentiating the following examples.

1. $y = 3x^2$. *Ans.* $\frac{dy}{dx} = 6x$.

2. $y = x^2 - 3x$. $\frac{dy}{dx} = 2x - 3$.

3. $y = ax^2 + bx + c$. $\frac{dy}{dx} = 2ax + b$.

4. $y = x^3$. $\frac{dy}{dx} = 3x^2$.

5. $r = a\theta^2$. $\frac{dr}{d\theta} = 2a\theta$.

6. $p = 2q^2$. $\frac{dp}{dq} = 4q$.

7. $s = t^2 - 2t + 3$. $\frac{ds}{dt} = 2t - 2$.

8. $y = \frac{1}{x}$. $\frac{dy}{dx} = -\frac{1}{x^2}$.

9. $s = \frac{2}{t^2}$. $\frac{ds}{dt} = -\frac{4}{t^3}$.

10. Find the slope of the tangent to the curve $y = 2x^3 - 6x + 5$, (a) at the point where $x = 1$; (b) at the point where $x = 0$. *Ans.* (a) 0; (b) $-6$.

11. (a) Find the slopes of the tangents to the two curves $y = 3x^2 - 1$ and $y = 2x^2 + 3$ at their points of intersection. (b) At what angle do they intersect? *Ans.* (a) $\pm 12$, $\pm 8$; (b) arc tan $\frac{4}{97}$.

# CHAPTER VI

## RULES FOR DIFFERENTIATING STANDARD ELEMENTARY FORMS

**46. Importance of General Rule.** The *General Rule* for differentiation given in the last chapter, p. 42, is fundamental, being found directly from the definition of a derivative, and it is very important that the student should be thoroughly familiar with it. However, the process of applying the rule to examples in general has been found too tedious or difficult; consequently special rules have been derived from the *General Rule* for differentiating certain standard forms of frequent occurrence in order to facilitate the work.

It has been found convenient to express these special rules by means of formulas, a list of which follows. The student should not only memorize each formula when deduced, but should be able to state the corresponding rule in words.

In these formulas, $u$, $v$, and $w$ denote *variable* quantities which are functions of $x$, and are differentiable.

### FORMULAS FOR DIFFERENTIATION

I $$\frac{dc}{dx} = 0.$$

II $$\frac{dx}{dx} = 1.$$

III $$\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

IV $$\frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

V $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

VI $$\frac{d}{dx}(v_1v_2\cdots v_n) = (v_2v_3\cdots v_n)\frac{dv_1}{dx} + (v_1v_3\cdots v_n)\frac{dv_2}{dx} + \cdots + (v_1v_2\cdots v_{n-1})\frac{dv_n}{dx}.$$

**VII** $\frac{d}{dx}(v^n) = nv^{n-1}\frac{dv}{dx}.$

**VII a** $\frac{d}{dx}(x^n) = nx^{n-1}.$

**VIII** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

**VIII a** $\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{\frac{du}{dx}}{c}.$

**VIII b** $\frac{d}{dx}\left(\frac{c}{v}\right) = -\frac{c\frac{dv}{dx}}{v^2}.$

**IX** $\frac{d}{dx}(\log_a v) = \log_a e \cdot \frac{\frac{dv}{dx}}{v}.$

**IX a** $\frac{d}{dx}(\log v) = \frac{\frac{dv}{dx}}{v}.$

**X** $\frac{d}{dx}(a^v) = a^v \log a \frac{dv}{dx}.$

**X a** $\frac{d}{dx}(e^v) = e^v\frac{dv}{dx}.$

**XI** $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + \log u \cdot u^v\frac{dv}{dx}.$

**XII** $\frac{d}{dx}(\sin v) = \cos v\frac{dv}{dx}.$

**XIII** $\frac{d}{dx}(\cos v) = -\sin v\frac{dv}{dx}.$

**XIV** $\frac{d}{dx}(\tan v) = \sec^2 v\frac{dv}{dx}.$

**XV** $\frac{d}{dx}(\cot v) = -\csc^2 v\frac{dv}{dx}.$

**XVI** $\frac{d}{dx}(\sec v) = \sec v \tan v\frac{dv}{dx}.$

**XVII** $\frac{d}{dx}(\csc v) = -\csc v \cot v\frac{dv}{dx}.$

XVIII $$\frac{d}{dx}(\text{vers}\, v) = \sin v \frac{dv}{dx}.$$

XIX $$\frac{d}{dx}(\text{arc} \sin v) = \frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}.$$

XX $$\frac{d}{dx}(\text{arc} \cos v) = -\frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}.$$

XXI $$\frac{d}{dx}(\text{arc} \tan v) = \frac{\frac{dv}{dx}}{1 + v^2}.$$

XXII $$\frac{d}{dx}(\text{arc} \cot v) = -\frac{\frac{dv}{dx}}{1 + v^2}.$$

XXIII $$\frac{d}{dx}(\text{arc} \sec v) = \frac{\frac{dv}{dx}}{v\sqrt{v^2 - 1}}.$$

XXIV $$\frac{d}{dx}(\text{arc} \csc v) = -\frac{\frac{dv}{dx}}{v\sqrt{v^2 - 1}}.$$

XXV $$\frac{d}{dx}(\text{arc vers}\, v) = \frac{\frac{dv}{dx}}{\sqrt{2v - v^2}}.$$

XXVI $$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}, \ y \text{ being a function of } v.$$

XXVII $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \ y \text{ being a function of } x.$$

**47. Differentiation of a constant.** A function that is known to have the same value for every value of the independent variable is constant, and we may denote it by

$$y = c.$$

As $x$ takes on an increment $\Delta x$, the function does not change in value, that is, $\Delta y = 0$, and

$$\frac{\Delta y}{\Delta x} = 0.$$

But $$\lim_{\Delta x = 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = 0.$$

I $$\therefore\ \frac{dc}{dx} = 0.$$

*The derivative of a constant is zero.*

**48. Differentiation of a variable with respect to itself.**

Let $$y = x.$$

Following *General Rule*, p. 42, we have

*First step.* $$y + \Delta y = x + \Delta x.$$

*Second step.* $$\Delta y = \Delta x.$$

*Third step.* $$\frac{\Delta y}{\Delta x} = 1.$$

*Fourth step.* $$\frac{dy}{dx} = 1.$$

II $$\therefore\ \frac{dx}{dx} = 1.$$

*The derivative of a variable with respect to itself is unity.*

**49. Differentiation of a sum.**

Let $$y = u + v - w.$$

By *General Rule*,

*First step.* $$y + \Delta y = u + \Delta u + v + \Delta v - w - \Delta w.$$

*Second step.* $$\Delta y = \Delta u + \Delta v - \Delta w.$$

*Third step.* $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x}.$$

*Fourth step.* $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

[Applying Th. I, p. 27.]

III $$\therefore\ \frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

Similarly for the algebraic sum of any finite number of functions.

*The derivative of the algebraic sum of a finite number of functions is equal to the same algebraic sum of their derivatives.*

**50. Differentiation of the product of a constant and a variable.**

Let $$y = cv.$$

By *General Rule*,

*First step.* $$y + \Delta y = c(v + \Delta v) = cv + c\Delta v.$$

*Second step.* $$\Delta y = c \cdot \Delta v.$$

*Third step.* $$\frac{\Delta y}{\Delta x} = c\frac{\Delta v}{\Delta x}.$$

*Fourth step.* $$\frac{dy}{dx} = c\frac{dv}{dx}.$$

[Applying Th. II, p. 27.]

IV $$\therefore \quad \frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

*The derivative of the product of a constant and a variable is equal to the product of the constant and the derivative of the variable.*

**51. Differentiation of the product of two variables.**

Let $$y = uv.$$

By *General Rule*,

*First step.* $$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v. \end{aligned}$$

*Second step.* $$\Delta y = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v.$$

*Third step.* $$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

*Fourth step.* $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

[Applying Th. II, p. 27, since when $\Delta x$ approaches zero as a limit, $\Delta u$ also approaches zero as a limit, and limit $\left(\Delta u\frac{\Delta v}{\Delta x}\right) = 0$.]

V $$\therefore \quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

*The derivative of the product of two variables is equal to the first variable times the derivative of the second, plus the second variable times the derivative of the first.*

**52. Differentiation of the product of any finite number of variables.** By dividing both sides of V by $uv$, it assumes the form

$$\frac{\frac{d}{dx}(uv)}{uv} = \frac{\frac{du}{dx}}{u} + \frac{\frac{dv}{dx}}{v}.$$

If then we have the product of $n$ variables

$$y = v_1 v_2 \cdots v_n,$$

we may write

$$\frac{\frac{d}{dx}(v_1 v_2 \cdots v_n)}{v_1 v_2 \cdots v_n} = \frac{\frac{dv_1}{dx}}{v_1} + \frac{\frac{d}{dx}(v_2 v_3 \cdots v_n)}{v_2 v_3 \cdots v_n}$$

$$= \frac{\frac{dv_1}{dx}}{v_1} + \frac{\frac{dv_2}{dx}}{v_2} + \frac{\frac{d}{dx}(v_3 v_4 \cdots v_n)}{v_3 v_4 \cdots v_n}$$

$$= \frac{\frac{dv_1}{dx}}{v_1} + \frac{\frac{dv_2}{dx}}{v_2} + \frac{\frac{dv_3}{dx}}{v_3} + \cdots + \frac{\frac{dv_n}{dx}}{v_n}.$$

Multiplying both sides by $v_1 v_2 \cdots v_n$, we get

**VI** $$\frac{d}{dx}(v_1 v_2 \cdots v_n) = (v_2 v_3 \cdots v_n)\frac{dv_1}{dx} + (v_1 v_3 \cdots v_n)\frac{dv_2}{dx} + \cdots + (v_1 v_2 \cdots v_{n-1})\frac{dv_n}{dx}.$$

*The derivative of the product of a finite number of variables is equal to the sum of all the products that can be formed by multiplying the derivative of each variable by all the other variables.*

**53. Differentiation of a variable with a constant exponent.** If the $n$ factors in VI are each equal to $v$, we get

$$\frac{\frac{d}{dx}(v^n)}{v^n} = n\frac{\frac{dv}{dx}}{v}.$$

**VII** $$\therefore \quad \frac{d}{dx}(v^n) = nv^{n-1}\frac{dv}{dx}.$$

When $v = x$ this becomes

**VII *a*** $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

We have so far proven VII only for the case when $n$ is a positive integer. In § 59, however, it will be shown that this formula holds true for any value of $n$ and we shall make use of this general result now.

*The derivative of a variable with a constant exponent is equal to the product of the exponent, the variable with the exponent diminished by unity, and the derivative of the variable.*

**54. Differentiation of a quotient.**

Let $$y = \frac{u}{v}, \quad v \neq 0.$$

By *General Rule,*

*First step.* $$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

*Second step.* $$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v \cdot \Delta u - u \cdot \Delta v}{v(v + \Delta v)}.$$

*Third step.* $$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

*Fourth step.* $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

[Applying Theorems II and III, p. 27.]

**VIII** $$\therefore \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

*The derivative of a fraction is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

When the denominator is constant, set $v = c$ in **VIII**, giving

**VIII *a*** $$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{\frac{du}{dx}}{c}.$$

$$\left[\text{Since } \frac{dv}{dx} = \frac{dc}{dx} = 0.\right]$$

We may also get **VIII *a*** from **IV** as follows:

$$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{1}{c}\frac{du}{dx} = \frac{\frac{du}{dx}}{c}.$$

*The derivative of the quotient of a variable by a constant is equal to the derivative of the variable divided by the constant.*

When the numerator is constant, set $u = c$ in VIII, giving

$$\textbf{VIII } b \qquad \frac{d}{dx}\left(\frac{c}{v}\right) = -\frac{c\frac{dv}{dx}}{v^2}.$$

$$\left[\text{Since } \frac{du}{dx} = \frac{dc}{dx} = 0.\right]$$

*The derivative of the quotient of a constant by a variable is equal to minus the product of the constant and the derivative of the variable, divided by the square of the variable.*

All explicit algebraic functions of one independent variable may be differentiated by following the rules we have deduced so far.

### EXAMPLES

Differentiate the following.

1. $y = x^3$.

*Solution.* $\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2$. *Ans.* by VII *a*

$[n = 3.]$

2. $y = ax^4 - bx^2$.

*Solution.* $\frac{dy}{dx} = \frac{d}{dx}(ax^4 - bx^2) = \frac{d}{dx}(ax^4) - \frac{d}{dx}(bx^2)$ by III

$= a\frac{d}{dx}(x^4) - b\frac{d}{dx}(x^2)$ by IV

$= 4ax^3 - 2bx$. *Ans.* by VII *a*

3. $y = x^{\frac{4}{3}} + 5$.

*Solution.* $\frac{dy}{dx} = \frac{d}{dx}(x^{\frac{4}{3}}) + \frac{d}{dx}(5)$ by III

$= \frac{4}{3}x^{\frac{1}{3}}$. *Ans.* by VII *a* and I

4. $y = \frac{3x^3}{\sqrt[5]{x^2}} - \frac{7x}{\sqrt[3]{x^4}} + 8\sqrt[7]{x^3}$.

*Solution.* $\frac{dy}{dx} = \frac{d}{dx}(3x^{\frac{13}{5}}) - \frac{d}{dx}(7x^{-\frac{1}{3}}) + \frac{d}{dx}(8x^{\frac{3}{7}})$ by III

$= \frac{39}{5}x^{\frac{8}{5}} + \frac{7}{3}x^{-\frac{4}{3}} + \frac{24}{7}x^{-\frac{4}{7}}$. *Ans.* by IV and VII *a*

5. $y = (x^2 - 3)^5$.

*Solution.* $\frac{dy}{dx} = 5(x^2 - 3)^4\frac{d}{dx}(x^2 - 3)$ by VII

$[v = x^2 - 3 \text{ and } n = 4.]$

$= 5(x^2 - 3)^4 \cdot 2x = 10x(x^2 - 3)^4$. *Ans.*

We might have expanded this function by the Binomial Theorem and then applied III, etc., but the above process is to be preferred.

6. $y = \sqrt{a^2 - x^2}$.

*Solution.* $\dfrac{dy}{dx} = \dfrac{d}{dx}(a^2 - x^2)^{\frac{1}{2}} = \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}\dfrac{d}{dx}(a^2 - x^2)$ by VII

[$v = a^2 - x^2$ and $n = \frac{1}{2}$.]

$$= \tfrac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{a^2 - x^2}}. \quad \textit{Ans.}$$

7. $y = (3x^2 + 2)\sqrt{1 + 5x^2}$.

*Solution.* $\dfrac{dy}{dx} = (3x^2 + 2)\dfrac{d}{dx}(1 + 5x^2)^{\frac{1}{2}} + (1 + 5x^2)^{\frac{1}{2}}\dfrac{d}{dx}(3x^2 + 2)$ by V

[$u = 3x^2 + 2$ and $v = (1 + 5x^2)^{\frac{1}{2}}$.]

$= (3x^2 + 2)\frac{1}{2}(1 + 5x^2)^{-\frac{1}{2}}\dfrac{d}{dx}(1 + 5x^2) + (1 + 5x^2)^{\frac{1}{2}}6x$ by VII, etc.

$$= (3x^2 + 2)(1 + 5x^2)^{-\frac{1}{2}}5x + 6x(1 + 5x^2)^{\frac{1}{2}}$$

$$= \frac{5x(3x^2 + 2)}{\sqrt{1 + 5x^2}} + 6x\sqrt{1 + 5x^2} = \frac{45x^3 + 16x}{\sqrt{1 + 5x^2}}. \quad \textit{Ans.}$$

8. $y = \dfrac{a^2 + x^2}{\sqrt{a^2 - x^2}}$.

*Solution.* $\dfrac{dy}{dx} = \dfrac{(a^2 - x^2)^{\frac{1}{2}}\dfrac{d}{dx}(a^2 + x^2) - (a^2 + x^2)\dfrac{d}{dx}(a^2 - x^2)^{\frac{1}{2}}}{a^2 - x^2}$ by VIII

$$= \frac{2x(a^2 - x^2) + x(a^2 + x^2)}{(a^2 - x^2)^{\frac{3}{2}}}$$

[Multiplying both numerator and denominator by $(a^2 - x^2)^{\frac{1}{2}}$.]

$$= \frac{3a^2x - x^3}{(a^2 - x^2)^{\frac{3}{2}}}. \quad \textit{Ans.}$$

9. $y = 5x^4 + 3x^2 - 6$. $\qquad \dfrac{dy}{dx} = 20x^3 + 6x$.

10. $y = 3cx^2 - 8dx + 5e$. $\qquad \dfrac{dy}{dx} = 6cx - 8d$.

11. $y = x^{a+b}$. $\qquad \dfrac{dy}{dx} = (a + b)x^{a+b-1}$.

12. $y = x^n + nx + n$. $\qquad \dfrac{dy}{dx} = nx^{n-1} + n$.

13. $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 + 5$. $\qquad f'(x) = 2x^2 - 3x$.

14. $f(x) = (a + b)x^2 + cx + d$. $\qquad f'(x) = 2(a + b)x + c$.

15. $\dfrac{d}{dx}(a + bx + cx^2) = b + 2cx$.

16. $\dfrac{d}{dy}(5y^m - 3y + 6) = 5my^{m-1} - 3$.

17. $v = \frac{v_0 p_0}{p}$. $\qquad \frac{dv}{dp} = -\frac{v_0 p_0}{p^2}$.

18. $v = v_0 + ft$. $\qquad \frac{dv}{dt} = f$.

19. $s = s_0 + v_0 t + \frac{1}{2} ft^2$. $\qquad \frac{ds}{dt} = v_0 + ft$.

20. $l = 1 + b\theta + c\theta^2$. $\qquad \frac{dl}{d\theta} = b + 2c\theta$.

21. $s = 2t^2 + 3t + 5$. $\qquad \frac{ds}{dt} = 4t + 3$.

22. $s = at^3 - bt^2 + c$. $\qquad \frac{ds}{dt} = 3at^2 - 2bt$.

23. $r = a\theta^2$. $\qquad \frac{dr}{d\theta} = 2a\theta$.

24. $r = c\theta^3 + d\theta^2 + e\theta$. $\qquad \frac{dr}{d\theta} = 3c\theta^2 + 2d\theta + e$.

25. $y = 6x^{\frac{7}{2}} + 4x^{\frac{5}{2}} + 2x^{\frac{3}{2}}$. $\qquad \frac{dy}{dx} = 21x^{\frac{5}{2}} + 10x^{\frac{3}{2}} + 3x^{\frac{1}{2}}$.

26. $y = \sqrt{3x} + \sqrt[3]{x} + \frac{1}{x}$. $\qquad \frac{dy}{dx} = \frac{3}{2\sqrt{3x}} + \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{x^2}$.

27. $y = \frac{a + bx + cx^2}{x}$. $\qquad \frac{dy}{dx} = c - \frac{a}{x^2}$.

28. $y = \frac{(x-1)^3}{x^{\frac{1}{3}}}$. $\qquad \frac{dy}{dx} = \frac{8}{3}x^{\frac{5}{3}} - 5x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}}$.

29. $y = \frac{x^{\frac{5}{2}} - x - x^{\frac{1}{2}} + a}{x^{\frac{3}{2}}}$. $\qquad \frac{dy}{dx} = \frac{2x^{\frac{5}{2}} + x + 2x^{\frac{1}{2}} - 3a}{2x^{\frac{5}{2}}}$.

30. $y = (2x^3 + x^2 - 5)^3$. $\qquad \frac{dy}{dx} = 6x(3x+1)(2x^3 + x^2 - 5)^2$.

31. $f(x) = (a + bx^2)^{\frac{5}{4}}$. $\qquad f'(x) = \frac{5bx}{2}(a + bx^2)^{\frac{1}{4}}$.

32. $f(x) = (1 + 4x^3)(1 + 2x^2)$. $\qquad f'(x) = 4x(1 + 3x + 10x^3)$.

33. $f(x) = (a + x)\sqrt{a - x}$. $\qquad f'(x) = \frac{a - 3x}{2\sqrt{a - x}}$.

34. $f(x) = (a + x)^m (b + x)^n$. $\qquad f'(x) = (a + x)^m (b + x)^n \left[\frac{m}{a + x} + \frac{n}{b + x}\right]$.

35. $y = \frac{1}{x^n}$. $\qquad \frac{dy}{dx} = -\frac{n}{x^{n+1}}$.

36. $y = x(a^2 + x^2)\sqrt{a^2 - x^2}$. $\qquad \frac{dy}{dx} = \frac{a^4 + a^2x^2 - 4x^4}{\sqrt{a^2 - x^2}}$.

37. $y = \dfrac{2x^4}{b^2 - x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{8b^2x^3 - 4x^5}{(b^2 - x^2)^2}$.

38. $y = \dfrac{a - x}{a + x}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2a}{(a + x)^2}$.

39. $s = \dfrac{t^3}{(1 + t)^2}$. $\qquad \dfrac{ds}{dt} = \dfrac{3t^2 + t^3}{(1 + t)^3}$.

40. $f(s) = \dfrac{(s + 4)^2}{s + 3}$. $\qquad f'(s) = \dfrac{(s + 2)(s + 4)}{(s + 3)^2}$.

41. $f(\theta) = \dfrac{\theta}{\sqrt{a - b\theta^2}}$. $\qquad f'(\theta) = \dfrac{a}{(a - b\theta^2)^{\frac{3}{2}}}$.

42. $F(r) = \sqrt{\dfrac{1 + r}{1 - r}}$. $\qquad F'(r) = \dfrac{1}{(1 - r)\sqrt{1 - r^2}}$.

43. $\psi(y) = \left(\dfrac{y}{1 - y}\right)^m$. $\qquad \psi'(y) = \dfrac{my^{m-1}}{(1 - y)^{m+1}}$.

44. $\phi(x) = \dfrac{2x^2 - 1}{x\sqrt{1 + x^2}}$. $\qquad \phi'(x) = \dfrac{1 + 4x^2}{x^2(1 + x^2)^{\frac{3}{2}}}$.

45. $$\frac{d}{dx}\left[\frac{1}{(a + x)^m(b + x)^n}\right] = -\frac{m(b + x) + n(a + x)}{(a + x)^{m+1}(b + x)^{n+1}}.$$

46. $$\frac{d}{dx}\left[\frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}\right] = -\frac{a^2 + a\sqrt{a^2 - x^2}}{x^2\sqrt{a^2 - x^2}}.$$

*Hint.* Rationalize the denominator first.

47. $y = \sqrt{2px}$. $\qquad \dfrac{dy}{dx} = \dfrac{p}{y}$.

48. $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$. $\qquad \dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$.

49. $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$. $\qquad \dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$.

50. $r = \sqrt{a\phi} + c\sqrt{\phi^3}$. $\qquad \dfrac{dr}{d\phi} = \dfrac{\sqrt{a} + 3c\phi}{2\sqrt{\phi}}$.

51. $u = \dfrac{v^c + v^d}{cd}$. $\qquad \dfrac{du}{dv} = \dfrac{v^{c-1}}{d} + \dfrac{v^{d-1}}{c}$.

52. $p = \dfrac{(q + 1)^{\frac{3}{2}}}{\sqrt{q - 1}}$. $\qquad \dfrac{dp}{dq} = \dfrac{(q - 2)\sqrt{q + 1}}{(q - 1)^{\frac{3}{2}}}$.

53. $y = \left[\dfrac{x}{1 + \sqrt{1 - x^2}}\right]^n$. $\qquad \dfrac{dy}{dx} = \dfrac{ny}{x\sqrt{1 - x^2}}$.

**54.** Given $(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$; find $(a+x)^4$ by differentiation.

**55.** Assuming that

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 \cdots + x^n,$$

deduce by differentiation the sum of the series

$$1 + 2x + 3x^2 + \cdots + nx^{n-1},$$

$n$ being any positive integer.

$$\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}. \quad \textit{Ans.}$$

**55. Differentiation of a function of a function.** It sometimes happens that $y$, instead of being defined directly as a function of $x$, is given as a function of another variable $v$ which is defined as a function of $x$. In that case $y$ is a function of $x$ through $v$ and is called a *function of a function.*

For example, if $\qquad y = \dfrac{2v}{1-v^2},$

and $\qquad v = 1 - x^2,$

then $y$ is a function of a function. By eliminating $v$ we may express $y$ directly as a function of $x$, but in general this is not the best plan when we wish to find $\dfrac{dy}{dx}$.

If $\qquad y = f(v),$

and $\qquad v = \phi(x),$

then $y$ is a function of $x$ through $v$. Let $x$ take on an increment $\Delta x$, giving

$$v + \Delta v = \phi(x + \Delta x), \text{ defining } \Delta v,$$

and

$$y + \Delta y = f(v + \Delta v), \text{ defining } \Delta y.$$

By multiplying both numerator and denominator of $\dfrac{\Delta y}{\Delta x}$ by $\Delta v$ we get

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta v} \cdot \frac{\Delta v}{\Delta x}.$$

Let $\Delta x$ approach the limit zero, then $\Delta v$ also approaches the limit zero, and we have,* applying Th. II, p. 27,

$$(A) \qquad \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

---

* Assuming that $\Delta v \neq 0$ for $\Delta x$ sufficiently small but not zero.

This may also be written

$$\text{(B)} \qquad \frac{dy}{dx} = f'(v)\cdot\phi'(x).$$

*If $y = f(v)$ and $v = \phi(x)$, the derivative of $y$ with respect to $x$ equals the product of the derivative of $y$ with respect to $v$ and the derivative of $v$ with respect to $x$.*[*]

**56. Differentiation of inverse functions.** Let

$$\text{(A)} \qquad x = \phi(y).$$

If the inverse function exists, denote it by

$$\text{(B)} \qquad y = f(x).$$

Differentiating (*B*) with respect to $y$ gives

$$1 = f'(x)\frac{dx}{dy}, \qquad \text{by } (B),\ \S\ 55$$

[Assuming $\phi(y)$ and $f(x)$ to be differentiable.]

or,

$$1 = \frac{dy}{dx}\cdot\frac{dx}{dy}.$$

If then $\frac{dx}{dy} = \phi'(y)$ is different from zero, we get

$$\text{(C)} \qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},$$

or,

$$\text{(D)} \qquad f'(x) = \frac{1}{\phi'(y)}.$$

*The derivative of the inverse function is equal to the reciprocal of the derivative of the direct function.*

**57. Differentiation of a logarithm.**

Let $$y = \log_a v.\dagger$$

Differentiating by the *General Rule*, p. 42, considering $v$ as the independent variable, we have

*First step.* $$y + \Delta y = \log_a (v + \Delta v).$$

---

[*] It is understood that $y$ and $v$ have fixed initial values corresponding to some fixed initial value of $x$.

[†] The student must not forget that this function is defined only for positive values of the base $a$ and the variable $v$.

*Second step.* $\Delta y = \log_a(v+\Delta v) - \log_a v$

$$= \log_a\left(\frac{v+\Delta v}{v}\right) = \log_a\left(1+\frac{\Delta v}{v}\right).$$

[By 8, p. 2.]

*Third step.* $\dfrac{\Delta y}{\Delta v} = \dfrac{1}{\Delta v}\log_a\left(1+\dfrac{\Delta v}{v}\right) = \log_a\left(1+\dfrac{\Delta v}{v}\right)^{\frac{1}{\Delta v}}$

$$= \frac{1}{v}\log_a\left(1+\frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}.$$

[Dividing the logarithm by $v$ and at the same time multiplying the exponent of the parenthesis by $v$ changes the form of the expression but not its value (see 9, p. 2).]

*Fourth step.* $\dfrac{dy}{dv} = \dfrac{1}{v}\log_a e.$*

[When $\Delta v$ approaches the limit zero, $\frac{\Delta v}{v}$ also approaches the limit zero.

Therefore $\underset{\Delta v=0}{\text{limit}}\left(1+\frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}} = e$, from Theorem II, p. 33, placing $a = \frac{\Delta v}{v}$.]

Hence

(*A*) $$\frac{dy}{dv} = \frac{d}{dv}\left(\log_a v\right) = \log_a e \cdot \frac{1}{v}.$$

Since $v$ is a function of $x$ and it is required to differentiate $\log_a v$ with respect to $x$, we must use formula (*A*), § 55, for differentiating a *function of a function*, namely,

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}.$$

Substituting value of $\frac{dy}{dv}$ from (*A*), we get

$$\frac{dy}{dx} = \log_a e \cdot \frac{1}{v}\cdot\frac{dv}{dx}.$$

**IX** $$\therefore \quad \frac{d}{dx}(\log_a v) = \log_a e \cdot \frac{\frac{dv}{dx}}{v}.$$

When $a = e$ this becomes

**IX *a*** $$\frac{d}{dx}(\log v) = \frac{\frac{dv}{dx}}{v}.$$

---

* Since $\log_a v$ is a continuous function.

*The derivative of the logarithm of a variable is equal to the product of the modulus*[*] *of the system of logarithms and the derivative of the variable, divided by the variable.*

**58. Differentiation of the simple exponential function.**

Let $$y = a^v. \qquad a > 0$$

Taking the logarithm of both sides to the base $e$, we get

$$\log y = v \log a,$$

or, $$v = \frac{\log y}{\log a} = \frac{1}{\log a} \cdot \log y.$$

Differentiate with respect to $y$ by formula **II** $\boldsymbol{a}$,

$$\frac{dv}{dy} = \frac{1}{\log a} \cdot \frac{1}{y};$$

and from ($C$), § 56, relating to *inverse functions*, we get

$$\frac{dy}{dv} = \log a \cdot y, \text{ or,}$$

$$\frac{dy}{dv} = \log a \cdot a^v. \tag{A}$$

Since $v$ is a function of $x$ and it is required to differentiate $a^v$ with respect to $x$, we must use formula ($A$), § 55, for differentiating a *function of a function*, namely,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

Substituting the value of $\frac{dy}{dv}$ from ($A$), we get

$$\frac{dy}{dx} = \log a \cdot a^v \cdot \frac{dv}{dx}.$$

$$\therefore \quad \boldsymbol{\frac{d}{dx}(a^v) = \log a \cdot a^v \cdot \frac{dv}{dx}}. \tag{I}$$

When $a = e$, this becomes

$$\boldsymbol{\frac{d}{dx}(e^v) = e^v \frac{dv}{dx}}. \tag{I a}$$

---

[*] The logarithm of $e$ to any base $a$ ($= \log_a e$) is called the *modulus* of the system whose base is $a$. In Algebra it is shown that we may find the logarithm of a number $N$ to any base $a$ by means of the formula

$$\log_a N = \log_a e \cdot \log_e N = \frac{\log_e N}{\log_e a}.$$

The modulus of the common or Briggs' system with base 10 is

$$\log_{10} e = .434294 \cdots.$$

*The derivative of a constant with a variable exponent is equal to the product of the natural logarithm of the constant, the constant with the variable exponent, and the derivative of the exponent.*

**59. Differentiation of the general exponential function.**

Let $$y = u^v.^*$$

Taking the logarithm of both sides to the base $e$,

$$\log_e y = v \log_e u,$$

or, $$y = e^{v \log u}.$$

Differentiating by formula **I** *a*,

$$\frac{dy}{dx} = e^{v \log u} \frac{d}{dx}(v \log u)$$

$$= e^{v \log u}\left(v \frac{\frac{du}{dx}}{u} + \log u \frac{dv}{dx}\right) \qquad \text{by } \mathbf{V}$$

$$= u^v\left(v \frac{\frac{du}{dx}}{u} + \log u \frac{dv}{dx}\right).$$

$$\mathbf{XI} \qquad \therefore \quad \frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + \log u \cdot u^v \frac{dv}{dx}.$$

*The derivative of a variable with a variable exponent is equal to the sum of the two results obtained by first differentiating by* **VII**, *regarding the exponent as constant; and again differentiating by* **X**, *regarding the base as constant.*

Let $v = n$, any constant; then **XI** reduces to

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

But this is the form differentiated in § 53, therefore **VIII** holds true for any value of $n$.

**Ex. 1.** Differentiate $y = \log(x^2 + a)$.

*Solution.*
$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2 + a)}{x^2 + a} \qquad \text{by IX } a$$
$$[v = x^2 + a.]$$
$$= \frac{2x}{x^2 + a}. \quad \textit{Ans.}$$

---

* $u$ can here assume only positive values.

**Ex. 2.** Differentiate $y = \log\sqrt{1 - x^2}$.

*Solution.*

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(1 - x^2)^{\frac{1}{2}}}{(1 - x^2)^{\frac{1}{2}}} \qquad \text{by IX } a$$

$$= \frac{\frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)}{(1 - x^2)^{\frac{1}{2}}} \qquad \text{by VII}$$

$$= \frac{x}{x^2 - 1}. \quad \textit{Ans.}$$

**Ex. 3.** Differentiate $y = a^{3x^2}$.

*Solution.*

$$\frac{dy}{dx} = \log a \cdot a^{3x^2}\frac{d}{dx}(3x^2) \qquad \text{by X}$$

$$= 6x\log a \cdot a^{3x^2}. \quad \textit{Ans.}$$

**Ex. 4.** Differentiate $y = be^{c^2 + x^2}$.

*Solution.*

$$\frac{dy}{dx} = b\frac{d}{dx}(e^{c^2 + x^2}) \qquad \text{by IV}$$

$$= be^{c^2 + x^2}\frac{d}{dx}(c^2 + x^2) \qquad \text{by X } a$$

$$= 2bxe^{c^2 + x^2}. \quad \textit{Ans.}$$

**Ex. 5.** Differentiate $y = x^{e^x}$.

*Solution.*

$$\frac{dy}{dx} = e^x x^{e^x - 1}\frac{d}{dx}(x) + x^{e^x}\log x\frac{d}{dx}(e^x) \qquad \text{by XI}$$

$$= e^x x^{e^x - 1} + x^{e^x}\log x \cdot e^x$$

$$= e^x x^{e^x}\left(\frac{1}{x} + \log x\right). \quad \textit{Ans.}$$

**60. Logarithmic differentiation.** Instead of applying IX and IX $a$ at once in differentiating logarithmic functions, we may sometimes simplify the work by first making use of one of the formulas 7–10 on p. 2. Thus above Ex. 2 may be solved as follows.

**Ex. 1.** Differentiate $y = \log\sqrt{1 - x^2}$.

*Solution.* By using 10, p. 2, we may write this in a form free from radicals as follows.

$$y = \tfrac{1}{2}\log(1 - x^2).$$

Then

$$\frac{dy}{dx} = \frac{1}{2}\,\frac{\frac{d}{dx}(1 - x^2)}{1 - x^2} \qquad \text{by IX } a$$

$$= \frac{1}{2}\cdot\frac{-2x}{1 - x^2} = \frac{x}{x^2 - 1}. \quad \textit{Ans.}$$

**Ex. 2.** Differentiate $y = \log\sqrt{\dfrac{1+x^2}{1-x^2}}$.

*Solution.* Simplifying by means of 10 and 8, p. 2,

$$y = \tfrac{1}{2}[\log(1+x^2) - \log(1-x^2)].$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{\frac{d}{dx}(1+x^2)}{1+x^2} - \frac{\frac{d}{dx}(1-x^2)}{1-x^2}\right] \qquad \text{by IX}\,a\text{, etc.}$$

$$= \frac{x}{1+x^2} + \frac{x}{1-x^2} = \frac{2x}{1-x^4}. \quad \textit{Ans.}$$

In differentiating an exponential function, especially a variable with a variable exponent, the best plan is first to take the logarithm of the function and then differentiate. Thus Ex, 5, p. 62, is solved more elegantly as follows.

**Ex. 3.** Differentiate $y = x^{e^x}$.

*Solution.* Taking the logarithm of both sides,

$$\log y = e^x \log x. \qquad \text{By 9, p. 2}$$

Now differentiate both sides with respect to $x$.

$$\frac{\frac{dy}{dx}}{y} = e^x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(e^x) \qquad \text{by IX}\,a\text{ and V}$$

$$= e^x\cdot\frac{1}{x} + \log x\cdot e^x,$$

or,

$$\frac{dy}{dx} = e^x\cdot y\left(\frac{1}{x} + \log x\right)$$

$$= e^x x^{e^x}\left(\frac{1}{x} + \log x\right). \quad \textit{Ans.}$$

**Ex. 4.** Differentiate $y = (4x^2 - 7)^{2+\sqrt{x^2-5}}$.

*Solution.* Taking logarithm,

$$\log y = (2 + \sqrt{x^2-5})\log(4x^2 - 7).$$

Differentiating both sides with respect to $x$,

$$\frac{1}{y}\frac{dy}{dx} = (2+\sqrt{x^2-5})\frac{8x}{4x^2-7} + \log(4x^2-7)\cdot\frac{x}{\sqrt{x^2-5}}.$$

$$\frac{dy}{dx} = x(4x^2-7)^{2+\sqrt{x^2-5}}\left[\frac{8(2+\sqrt{x^2-5})}{4x^2-7} + \frac{\log(4x^2-7)}{\sqrt{x^2-5}}\right]. \quad \textit{Ans.}$$

In the case of a function consisting of a number of factors it is sometimes convenient to take the logarithm before differentiating. Thus,

**Ex. 5.** Differentiate $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}$.

*Solution.* Taking logarithm,

$$\log y = \tfrac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4)].$$

Differentiating both sides with respect to $x$,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}\right]$$

$$=-\frac{2x^2-10x+11}{(x-1)(x-2)(x-3)(x-4)},$$

or,

$$\frac{dy}{dx}=-\frac{2x^2-10x-11}{(x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}(x-3)^{\frac{3}{2}}(x-4)^{\frac{3}{2}}}. \quad \textit{Ans.}$$

## EXAMPLES

Differentiate the following.

1. $y=\log(x+a)$. $\qquad \frac{dy}{dx}=\frac{1}{x+a}$.

2. $y=\log(ax+b)$. $\qquad \frac{dy}{dx}=\frac{a}{ax+b}$.

3. $y=\log\frac{1+x}{1-x}$. $\qquad \frac{dy}{dx}=\frac{2}{1-x^2}$.

4. $y=\log\frac{1+x^2}{1-x^2}$. $\qquad \frac{dy}{dx}=\frac{4x}{1-x^4}$.

5. $y=e^{ax}$. $\qquad \frac{dy}{dx}=ae^{ax}$.

6. $y=e^{4x+5}$ $\qquad \frac{dy}{dx}=4e^{4x+5}$.

7. $y=\log(x^2+x)$. $\qquad \frac{dy}{dx}=\frac{2x+1}{x^2+x}$.

8. $y=\log(x^3-2x+5)$. $\qquad \frac{dy}{dx}=\frac{3x^2-2}{x^3-2x+5}$.

9. $y=\log_a(2x+x^3)$. $\qquad \frac{dy}{dx}=\log_a e\cdot\frac{2+3x^2}{2x+x^3}$.

10. $y=x\log x$. $\qquad \frac{dy}{dx}=\log x+1$.

11. $f(x)=\log x^3$. $\qquad f'(x)=\frac{3}{x}$.

12. $f(x)=\log^3 x$. $\qquad f'(x)=\frac{3\log^2 x}{x}$.

*Hint.* $\log^3 x=(\log x)^3$. Use first VII, $v=\log x$, $n=3$; and then IX $a$.

13. $f(x)=\log\frac{a+x}{a-x}$. $\qquad f'(x)=\frac{2a}{a^2-x^2}$.

14. $f(x)=\log(x+\sqrt{1+x^2})$. $\qquad f'(x)=\frac{1}{\sqrt{1+x^2}}$.

15. $y = a^{e^x}$. $\qquad \dfrac{dy}{dx} = \log a \cdot a^{e^x} e^x$.

16. $y = b^{x^2}$. $\qquad \dfrac{dy}{dx} = 2x \log b \cdot b^{x^2}$.

17. $y = 7^{x^2+2x}$. $\qquad \dfrac{dy}{dx} = 2 \log 7 \cdot (x + 1)\, 7^{x^2+2x}$.

18. $y = c^{a^2-x^2}$. $\qquad \dfrac{dy}{dx} = -2x \log c \cdot c^{a^2-x^2}$.

19. $r = a^{\theta}$. $\qquad \dfrac{dr}{d\theta} = a^{\theta} \log a$.

20. $r = a^{\log \theta}$. $\qquad \dfrac{dr}{d\theta} = \dfrac{a^{\log \theta} \log a}{\theta}$.

21. $s = e^{b^2+t^2}$. $\qquad \dfrac{ds}{dt} = 2te^{b^2+t^2}$.

22. $u = ae^{\sqrt{v}}$. $\qquad \dfrac{du}{dv} = \dfrac{ae^{\sqrt{v}}}{2\sqrt{v}}$.

23. $p = e^{q \log q}$. $\qquad \dfrac{dp}{dq} = e^{q \log q}(1 + \log q)$.

24. $\dfrac{d}{dx}[e^x(1 - x^2)] = e^x(1 - 2x - x^2)$.

25. $\dfrac{d}{dx}\left(\dfrac{e^x - 1}{e^x + 1}\right) = \dfrac{2e^x}{(e^x + 1)^2}$.

26. $\dfrac{d}{dx}(x^2e^{ax}) = xe^{ax}(ax + 2)$.

27. $y = \log \dfrac{e^x}{1 + e^x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{1 + e^x}$.

28. $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

29. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{4}{(e^x + e^{-x})^2}$.

30. $y = x^n a^x$. $\qquad \dfrac{dy}{dx} = a^x x^{n-1}(n + x \log a)$.

31. $y = x^x$. $\qquad \dfrac{dy}{dx} = x^x(\log x + 1)$.

32. $y = x^{\frac{1}{x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^{\frac{1}{x}}(1 - \log x)}{x^2}$.

33. $y = x^{\log x}$. $\qquad \dfrac{dy}{dx} = \log x^2 \cdot x^{\log x - 1}$.

34. $f(y) = \log y \cdot e^y.$ $\qquad f'(y) = e^y\left(\log y + \frac{1}{y}\right).$

35. $f(s) = \frac{\log s}{e^s}.$ $\qquad f'(s) = \frac{1 - s\log s}{se^s}.$

36. $f(x) = \log(\log x).$ $\qquad f'(x) = \frac{1}{x\log x}.$

37. $F(x) = \log^4(\log x).$ $\qquad F'(x) = \frac{4\log^3(\log x)}{x\log x}.$

38. $\phi(x) = \log(\log^4 x).$ $\qquad \phi'(x) = \frac{4}{x\log x}.$

39. $\psi(y) = \log\sqrt{\frac{1+y}{1-y}}.$ $\qquad \psi'(y) = \frac{1}{1-y^2}.$

40. $f(x) = \log\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}.$ $\qquad f'(x) = -\frac{2}{\sqrt{1+x^2}}.$

*Hint.* First rationalize the denominator.

41. $y = x^{\frac{1}{\log x}}.$ $\qquad \frac{dy}{dx} = 0.$

42. $y = e^{x^x}.$ $\qquad \frac{dy}{dx} = e^{x^x}(1 + \log x)x^x.$

43. $y = \frac{c^x}{x^x}.$ $\qquad \frac{dy}{dx} = \left(\frac{c}{x}\right)^x\left(\log\frac{c}{x} - 1\right).$

44. $y = \left(\frac{x}{n}\right)^{nx}.$ $\qquad \frac{dy}{dx} = n\left(\frac{x}{n}\right)^{nx}\left(1 + \log\frac{x}{n}\right).$

45. $w = v^{e^v}.$ $\qquad \frac{dw}{dv} = v^{e^v}e^v\left(\frac{1 + v\log v}{v}\right).$

46. $z = \left(\frac{a}{t}\right)^t.$ $\qquad \frac{dz}{dt} = \left(\frac{a}{t}\right)^t(\log a - \log t - 1).$

47. $y = x^{x^n}.$ $\qquad \frac{dy}{dx} = x^{x^n+n-1}(n\log x + 1).$

48. $y = x^{x^x}.$ $\qquad \frac{dy}{dx} = x^{x^x}x^x\left(\log x + \log^2 x + \frac{1}{x}\right).$

49. $y = a^{\frac{1}{\sqrt{a^2-x^2}}}.$ $\qquad \frac{dy}{dx} = \frac{xy\log a}{(a^2-x^2)^{\frac{3}{2}}}.$

50. $y = e^x(x^n - nx^{n-1} + n(n-1)x^{n-2} + \cdots).$ $\qquad \frac{dy}{dx} = e^x x^n.$

51. $y = \frac{(x+1)^2}{(x+2)^3(x+3)^4}.$ $\qquad \frac{dy}{dx} = \frac{(x+1)(5x^2+14x+5)}{(x+2)^4(x+3)^5}.$

*Hint.* Take logarithm of both sides before differentiating in this and the following examples.

52. $y = \dfrac{(x-1)^{\frac{3}{2}}}{(x-2)^{\frac{1}{2}}(x-3)^{\frac{7}{3}}}$. $\dfrac{dy}{dx} = -\dfrac{(x-1)^{\frac{1}{2}}(7x^2+30x-97)}{12(x-2)^{\frac{3}{2}}(x-3)^{\frac{10}{3}}}$.

53. $y = x\sqrt{1-x}(1+x)$. $\dfrac{dy}{dx} = \dfrac{2+x-5x^2}{2\sqrt{1-x}}$.

54. $y = \dfrac{x(1+x^2)}{\sqrt{1-x^2}}$. $\dfrac{dy}{dx} = \dfrac{1+3x^2-2x^4}{(1-x^2)^{\frac{3}{2}}}$.

55. $y = x^5(a+3x)^3(a-2x)^2$. $\dfrac{dy}{dx} = 5x^4(a+3x)^2(a-2x)(a^2+2ax-12x^2)$.

56. $y = \dfrac{\sqrt{(x+a)^3}}{\sqrt{x-a}}$. $\dfrac{dy}{dx} = \dfrac{(x-2a)\sqrt{x+a}}{(x-a)^{\frac{3}{2}}}$.

### 61. Differentiation of sin *v*.

Let $y = \sin v$.

By *General Rule*, p. 42, considering $v$ as the independent variable, we have

*First step.* $y + \Delta y = \sin(v + \Delta v)$.

*Second step.*

$$\begin{aligned}\Delta y &= \sin(v+\Delta v) - \sin v\\ &= 2\cos\left(v + \frac{\Delta v}{2}\right)\cdot\sin\frac{\Delta v}{2}.^*\end{aligned}$$

*Third step.*

$$\frac{\Delta y}{\Delta v} = \cos\left(v + \frac{\Delta v}{2}\right)\left(\frac{\sin\frac{\Delta v}{2}}{\frac{\Delta v}{2}}\right).$$

*Fourth step.*

$$\frac{dy}{dv} = \cos v.$$

$$\left[\begin{aligned}&\text{Since} \quad \lim_{\Delta v = 0}\left(\frac{\sin\frac{\Delta v}{2}}{\frac{\Delta v}{2}}\right) = 1 \text{ by (14), p. 30,}\\ &\text{and} \quad \lim_{\Delta v = 0}\cos\left(v + \frac{\Delta v}{2}\right) = \cos v.\end{aligned}\right]$$

---

[*] Let $A = v + \Delta v$ and $B = v$. Adding, $A + B = 2v + \Delta v$. Therefore $\frac{1}{2}(A+B) = v + \frac{\Delta v}{2}$.

$A = v + \Delta v$, $B = v$. Subtracting, $A - B = \Delta v$; $\frac{1}{2}(A-B) = \frac{\Delta v}{2}$.

Substituting these values of $A$, $B$, $\frac{1}{2}(A+B)$, $\frac{1}{2}(A-B)$ in terms of $v$ and $\Delta v$ in the formula from Trigonometry (42, p. 3),

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B),$$

we get

$$\sin(v+\Delta v) - \sin v = 2\cos\left(v + \frac{\Delta v}{2}\right)\sin\frac{\Delta v}{2}.$$

Since $v$ is a function of $x$ and it is required to differentiate $\sin v$ with respect to $x$, we must use formula $(A)$, § 55, for differentiating a *function of a function*, namely,

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}.$$

Substituting value $\frac{dy}{dv}$ from *Fourth step*, we get

$$\frac{dy}{dx}=\cos v\frac{dv}{dx}.$$

$$\text{XII} \qquad \therefore\ \frac{d}{dx}(\sin v)=\cos v\frac{dv}{dx}.$$

The statement of the corresponding rules will now be left to the student.

**62. Differentiation of $\cos v$.**

Let $y=\cos v.$

By 29, p. 2, this may be written

$$y=\sin\left(\frac{\pi}{2}-v\right).$$

Differentiating by formula XII,

$$\frac{dy}{dx}=\cos\left(\frac{\pi}{2}-v\right)\frac{d}{dx}\left(\frac{\pi}{2}-v\right)$$

$$=\cos\left(\frac{\pi}{2}-v\right)\left(-\frac{dv}{dx}\right)$$

$$=-\sin v\frac{dv}{dx}.$$

$\left[\text{Since } \cos\left(\frac{\pi}{2}-v\right)=\sin v, \text{ by 29, p. 2.}\right]$

$$\text{XIII} \qquad \therefore\ \frac{d}{dx}(\cos v)=-\sin v\frac{dv}{dx}.$$

**63. Differentiation of $\tan v$.**

Let $y=\tan v.$

By 27, p. 2, this may be written

$$y=\frac{\sin v}{\cos v}.$$

Differentiating by formula VIII,

$$\frac{dy}{dx}=\frac{\cos v\frac{d}{dx}(\sin v)-\sin v\frac{d}{dx}(\cos v)}{\cos^2 v}$$

$$=\frac{\cos^2 v\frac{dv}{dx}+\sin^2 v\frac{dv}{dx}}{\cos^2 v}$$

$$=\frac{\frac{dv}{dx}}{\cos^2 v}=\sec^2 v\frac{dv}{dx}.$$

**XIV** $$\therefore\quad \frac{d}{dx}(\tan v)=\sec^2 v\frac{dv}{dx}.$$

**64. Differentiation of cot $v$.**

Let $$y=\cot v.$$

By 27, p. 2, this may be written

$$y=\frac{1}{\tan v}.$$

Differentiating by formula VIII *b*,

$$\frac{dy}{dx}=-\frac{\frac{d}{dx}(\tan v)}{\tan^2 v}$$

$$=-\frac{\sec^2 v\frac{dv}{dx}}{\tan^2 v}=-\csc v^2\frac{dv}{dx}.$$

**XV** $$\therefore\quad \frac{d}{dx}(\cot v)=-\csc^2 v\frac{dv}{dx}.$$

**65. Differentiation of sec $v$.**

Let $$y=\sec v.$$

By 26, p. 2, this may be written

$$y=\frac{1}{\cos v}.$$

Differentiating by formula **VIII *b***,

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(\cos v)}{\cos^2 v}$$

$$= \frac{\sin v \frac{dv}{dx}}{\cos^2 v}$$

$$= \sec v \tan v \frac{dv}{dx}.$$

**XVI** $$\therefore \quad \frac{d}{dx}(\sec v) = \sec v \tan v \frac{dv}{dx}.$$

**66. Differentiation of csc *v*.**

Let $y = \csc v.$

By 26, p. 2, this may be written

$$y = \frac{1}{\sin v}.$$

Differentiating by formula **VIII *b***,

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}(\sin v)}{\sin^2 v}$$

$$= -\frac{\cos v \frac{dv}{dx}}{\sin^2 v}$$

$$= -\csc v \cot v \frac{dv}{dx}.$$

**XVII** $$\therefore \quad \frac{d}{dx}(\csc v) = -\csc v \cot v \frac{dv}{dx}.$$

**67. Differentiation of vers *v*.**

Let $y = \text{vers } v.$

By Trigonometry this may be written

$$y = 1 - \cos v.$$

Differentiating,

$$\frac{dy}{dx} = \sin v \frac{dv}{dx}.$$

**XVIII** $$\therefore \quad \frac{d}{dx}(\text{vers } v) = \sin v \frac{dv}{dx}.$$

In the derivation of our formulas so far it has been necessary to apply the *General Rule*, p. 42 (i.e. the four steps), only for the following:

III $\frac{d}{dx}(u+v-w)=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}\cdot$ Algebraic sum.

V $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}\cdot$ Product.

VIII $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\cdot$ Quotient.

IX $\frac{d}{dx}(\log_a v)=\log_a e\frac{\frac{dv}{dx}}{v}\cdot$ Logarithm.

XII $\frac{d}{dx}(\sin v)=\cos v\frac{dv}{dx}\cdot$ Sine.

XXVI $\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}\cdot$ Function of a function.

Not only do all the other formulas we have deduced depend on these, but all we shall deduce hereafter depend on them as well. Hence it follows that the derivation of the fundamental formulas for differentiation involves the calculation of only two limits of any difficulty, viz.,

$$\underset{v=0}{\text{limit}}\frac{\sin v}{v}=1 \qquad \text{by (14), p. 30}$$

and

$$\underset{v=0}{\text{limit}}(1+v)^{\frac{1}{v}}=e. \qquad \text{By Th. II, p. 33}$$

### EXAMPLES

Differentiate the following.

1. $y=\sin ax^2$.

$\frac{dy}{dx}=\cos ax^2\frac{d}{dx}(ax^2)$ by XII

$[v=ax^2.]$

$=2\,ax\cos ax^2.$

2. $y = \tan\sqrt{1-x}$.

$$\frac{dy}{dx} = \sec^2\sqrt{1-x}\,\frac{d}{dx}(1-x)^{\frac{1}{2}} \qquad \text{by XIV}$$

$$[v = \sqrt{1-x}.]$$

$$= \sec^2\sqrt{1-x}\cdot\tfrac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$= -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}.$$

3. $y = \cos^3 x$.

This may also be written

$$y = (\cos x)^3.$$

$$\frac{dy}{dx} = 3(\cos x)^2\frac{d}{dx}(\cos x) \qquad \text{by VII}$$

$$[v = \cos x \text{ and } n = 3.]$$

$$= 3\cos^2 x\,(-\sin x) \qquad \text{by XIII}$$

$$= 3\sin x\cos^2 x.$$

4. $y = \sin nx \sin^n x$.

$$\frac{dy}{dx} = \sin nx\,\frac{d}{dx}(\sin x)^n + \sin^n x\,\frac{d}{dx}(\sin nx) \qquad \text{by V}$$

$$[u = \sin nx \text{ and } v = \sin^n x.]$$

$$= \sin nx\cdot n(\sin x)^{n-1}\frac{d}{dx}(\sin x) + \sin^n x\cos nx\,\frac{d}{dx}(nx) \qquad \text{by VII and XII}$$

$$= n\sin nx\cdot\sin^{n-1}x\cos x + n\sin^n x\cos nx$$

$$= n\sin^{n-1}x(\sin nx\cos x + \cos nx\sin x)$$

$$= n\sin^{n-1}x\sin(n+1)x.$$

| | |
|---|---|
| 5. $y = \sec ax$. | $\frac{dy}{dx} = a\sec ax\tan ax$. |
| 6. $y = \tan(ax+b)$. | $\frac{dy}{dx} = a\sec^2(ax+b)$. |
| 7. $y = \sin^2 x$. | $\frac{dy}{dx} = \sin 2x$. |
| 8. $y = \cos^3 x^2$. | $\frac{dy}{dx} = 6x\cos^2 x^2\sin x^2$. |
| 9. $f(y) = \sin 2y\cos y$. | $f'(y) = 2\cos 2y\cos y$. |
| 10. $F(x) = \cot^2 5x$. | $F'(x) = 10\cot 5x\,\text{cosec}^2\,5x$. |
| 11. $F(\theta) = \tan\theta - \theta$. | $F'(\theta) = \tan^2\theta$. |
| 12. $f(\phi) = \phi\sin\phi + \cos\phi$. | $f'(\phi) = \phi\cos\phi$. |
| 13. $f(t) = \sin^3 t\cos t$. | $f'(t) = \sin^2 t(3\cos^2 t - \sin^2 t)$. |
| 14. $r = a\cos 2\theta$. | $\frac{dr}{d\theta} = -2a\sin 2\theta$. |

15. $r = a\sqrt{\cos 2\theta}$. $\qquad \frac{dr}{d\theta} = -\frac{a \sin 2\theta}{\sqrt{\cos 2\theta}}$.

16. $r = a(1 - \cos\theta)$. $\qquad \frac{dr}{d\theta} = a\sin\theta$.

17. $r = a\sin^3\frac{\theta}{3}$. $\qquad \frac{dr}{d\theta} = a\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$.

18. $\frac{d}{dx}(\log\cos x) = -\tan x$.

19. $\frac{d}{dx}(\log\tan x) = \frac{2}{\sin 2x}$.

20. $\frac{d}{dx}(\log\sin^2 x) = 2\cot x$.

21. $y = \frac{\tan x - 1}{\sec x}$. $\qquad \frac{dy}{dx} = \sin x + \cos x$.

22. $y = \log\sqrt{\frac{1+\sin x}{1-\sin x}}$. $\qquad \frac{dy}{dx} = \frac{1}{\cos x}$.

23. $y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$. $\qquad \frac{dy}{dx} = \frac{1}{\cos x}$.

24. $f(x) = \sin(x+a)\cos(x-a)$. $\qquad f'(x) = \cos 2x$.

25. $f(x) = \sin(\log x)$. $\qquad f'(x) = \frac{\cos(\log x)}{x}$.

26. $f(x) = \tan(\log x)$. $\qquad f'(x) = \frac{\sec^2(\log x)}{x}$.

27. $s = \cos\frac{a}{t}$. $\qquad \frac{ds}{dt} = \frac{a\sin\frac{a}{t}}{t^2}$.

28. $r = \sin\frac{1}{\theta^2}$. $\qquad \frac{dr}{d\theta} = -\frac{2\cos\frac{1}{\theta^2}}{\theta^3}$.

29. $p = \sin(\cos q)$. $\qquad \frac{dp}{dq} = -\sin q\cos(\cos q)$.

30. $y = e^{\sin x}$. $\qquad \frac{dy}{dx} = e^{\sin x}\cos x$.

31. $y = a^{\tan nx}$. $\qquad \frac{dy}{dx} = na^{\tan nx}\sec^2 nx\log a$.

32. $y = e^{\cos x}\sin x$. $\qquad \frac{dy}{dx} = e^{\cos x}(\cos x - \sin^2 x)$.

33. $y = e^x\log\sin x$. $\qquad \frac{dy}{dx} = e^x(\cot x + \log\sin x)$.

34. $\frac{d}{dx}(x^n e^{\sin x}) = x^{n-1} e^{\sin x}(n + x \cos x).$

35. $\frac{d}{dx}(e^{ax} \cos mx) = e^{ax}(a \cos mx - m \sin mx).$

36. $f(\theta) = \frac{1 + \cos \theta}{1 - \cos \theta}.$ $\qquad f'(\theta) = -\frac{2 \sin \theta}{(1 - \cos \theta)^2}.$

37. $f(\phi) = \frac{e^{a\phi}(a \sin \phi - \cos \phi)}{a^2 + 1}.$ $\qquad f'(\phi) = e^{a\phi} \sin \phi.$

38. $f(s) = (s \cot s)^2.$ $\qquad f'(s) = 2 s \cot s (\cot s - s \operatorname{cosec}^2 s).$

39. $r = \frac{1}{3} \tan^3 \theta - \tan \theta + \theta.$ $\qquad \frac{dr}{d\theta} = \tan^4 \theta.$

40. $y = x^{\sin x}.$ $\qquad \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right).$

41. $y = (\sin x)^x.$ $\qquad \frac{dy}{dx} = (\sin x)^x [\log \sin x + x \cot x].$

42. $y = (\sin x)^{\tan x}.$ $\qquad \frac{dy}{dx} = (\sin x)^{\tan x}(1 + \sec^2 x \log \sin x).$

43. $y = x + \log \cos\left(x - \frac{\pi}{4}\right).$ $\qquad \frac{dy}{dx} = \frac{2}{1 + \tan x}.$

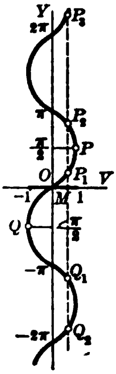
44. From $\sin 2x = 2 \sin x \cos x$, deduce by differentiation

$$\cos 2x = \cos^2 x - \sin^2 x.$$

45. From $\sin x + \sin 2x + \cdots + \sin nx = \dfrac{\sin \frac{n+1}{2} x \sin \frac{nx}{2}}{\sin \frac{x}{2}}$, deduce by differentiation

$$\cos x + 2 \cos 2x + \cdots + n \cos nx = \frac{\frac{n+1}{2} \sin \frac{x}{2} \sin \frac{2n+1}{2} x - \frac{1}{2}\left(\sin \frac{n+1}{2} x\right)^2}{\sin^2 \frac{x}{2}}.$$

[$n$ = a positive integer.]



## 68. Differentiation of arc sin $v$.

Let $\qquad y = \text{arc} \sin v;$*

then $\qquad v = \sin y.$

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* It should be remembered that this function is defined only for values of $v$ between $-1$ and $+1$ inclusive and that $y$ (the function) is many-valued, there being infinitely many arcs whose sines all equal $v$. Thus, in the figure (the locus of $y = \text{arc} \sin v$), when $v = OM$, $y = MP_1, MP_2, MP_3, \cdots, MQ_1, MQ_2, \cdots$. In the above discussion, in order to make the function single-valued, only values of $y$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ inclusive (points on arc $QOP$) are considered; that is, the arc of smallest numerical value whose sine is $v$.

Differentiating with respect to $y$ by XII,

$$\frac{dv}{dy} = \cos y;$$

therefore $$\frac{dy}{dv} = \frac{1}{\cos y}. \qquad \text{By } (C), \text{ p. } 58$$

But since $v$ is a function of $x$, this may be substituted in

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}, \qquad (A), \text{ p. } 57$$

giving $$\frac{dy}{dx} = \frac{1}{\cos y} \cdot \frac{dv}{dx}$$

$$= \frac{1}{\sqrt{1 - v^2}} \frac{dv}{dx}.$$

[$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - v^2}$; the positive sign of the radical being taken since $\cos y$ is positive for all values of $y$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ inclusive.]

XIX $$\therefore \frac{d}{dx}(\text{arc} \sin v) = \frac{\frac{dv}{dx}}{\sqrt{1 - v^2}}.$$

**69. Differentiation of arc cos $v$.**

Let $$y = \text{arc} \cos v;\text{*}$$

then $$v = \cos y.$$

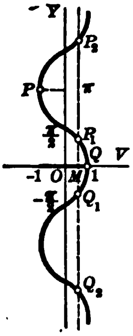
Differentiating with respect to $y$ by XIII,

$$\frac{dv}{dy} = -\sin y;$$

therefore $$\frac{dy}{dv} = -\frac{1}{\sin y}. \qquad \text{By } (C), \text{ p. } 58$$

But since $v$ is a function of $x$, this may be substituted in the formula

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}, \qquad (A), \text{ p. } 57$$



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[*] This function is defined only for values of $v$ between $-1$ and $+1$ inclusive, and is many-valued. In the figure (the locus of $y = \text{arc} \cos v$), when $v = OM$, $y = MP_1, MP_2, \cdots, MQ_1, MQ_2, \cdots$.

In order to make the function single-valued, only values of $y$ between 0 and $\pi$ inclusive are considered; that is, the smallest positive arc whose cosine is $v$. Hence we confine ourselves to arc $QP$ of the graph.

giving
$$\frac{dy}{dx}=-\frac{1}{\sin y}\cdot\frac{dv}{dx}$$
$$=-\frac{1}{\sqrt{1-v^2}}\frac{dv}{dx}.$$

$\left[\begin{array}{l}\sin y=\sqrt{1-\cos^2 y}=\sqrt{1-v^2}\text{, the plus sign of the radical being taken}\\ \text{since } \sin y \text{ is positive for all values of } y \text{ between 0 and } \pi \text{ inclusive.}\end{array}\right]$

**XI** $$\therefore\ \frac{d}{dx}(\text{arc}\cos v)=-\frac{\frac{dv}{dx}}{\sqrt{1-v^2}}.$$

### 70. Differentiation of arc tan $v$.

Let $y = \text{arc}\tan v$;*

then $v = \tan y$.

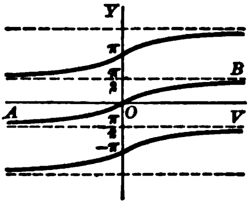
Differentiating with respect to $y$ by **XIV**,
$$\frac{dv}{dy}=\sec^2 y;$$

therefore
$$\frac{dy}{dv}=\frac{1}{\sec^2 y}.$$
By (*C*), p. 58

But since $v$ is a function of $x$, this may be substituted in the formula
$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx},$$
(*A*), p. 57

giving
$$\frac{dy}{dx}=\frac{1}{\sec^2 y}\cdot\frac{dv}{dx}$$
$$=\frac{1}{1+v^2}\frac{dv}{dx}.$$

$[\sec^2 y = 1+\tan^2 y = 1+v^2.]$



**XXI** $$\therefore\ \frac{d}{dx}(\text{arc}\tan v)=\frac{\frac{dv}{dx}}{1+v^2}.$$

---

* This function is defined for all values of $v$ and is many-valued, as is clearly shown by its graph. In order to make it single-valued, only values of $y$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are considered; that is, the arc of smallest numerical value whose tangent is $v$ (branch *AOB*).

### 71. Differentiation of arc cot $v$.*

Following the method of last section, we get

$$\text{XXIII} \qquad \frac{d}{dx}(\text{arc cot } v) = -\frac{\dfrac{dv}{dx}}{1+v^2}.$$

### 72. Differentiation of arc sec $v$.

Let $\qquad y = \text{arc sec } v;$†

then $\qquad v = \sec y.$

Differentiating with respect to $y$ by XVI,

$$\frac{dv}{dy} = \sec y \tan y;$$

therefore $\qquad \dfrac{dy}{dv} = \dfrac{1}{\sec y \tan y}. \qquad$ By (*C*), p. 58

But since $v$ is a function of $x$, this may be substituted in the formula

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dv}{dx}, \qquad (A), \text{ p. } 57$$

---

* This function is defined for all values of $v$ and is many-valued, as is seen from its graph (Fig. *a*). In order to make it single-valued, only values of $y$ between 0 and $\pi$ are considered; that is, the smallest positive arc whose cotangent is $v$. Hence we confine ourselves to branch *AB*.

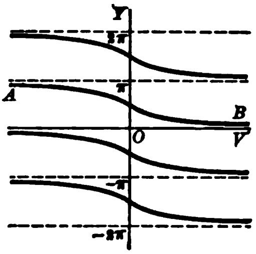


Fig. *a*

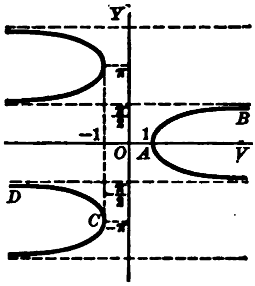


Fig. *b*

† This function is defined for all values of $v$ except those lying between −1 and +1, and is seen to be many-valued. To make the function single-valued, $y$ is taken as the arc of smallest numerical value whose secant is $v$. This means that if $v$ is positive we confine ourselves to points on arc *AB* (Fig. *b*), $y$ taking on values between 0 and $\frac{\pi}{2}$ (0 may be included); and if $v$ is negative we confine ourselves to points on arc *DC*, $y$ taking on values between $-\pi$ and $-\frac{\pi}{2}$ ($-\pi$ may be included).

giving
$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}\frac{dv}{dx}$$
$$= \frac{1}{v\sqrt{v^2-1}}\frac{dv}{dx}.$$

$\left[\sec y = v, \text{ and } \tan y = \sqrt{\sec^2 y - 1} = \sqrt{v^2-1},\right.$ the plus sign of the radical being taken since $\tan y$ is positive for all values of $y$ between 0 and $\frac{\pi}{2}$ and between $-\pi$ and $-\frac{\pi}{2}$, including 0 and $\left.-\pi.\right]$

XXIII
$$\therefore \quad \frac{d}{dx}(\text{arc} \sec v) = \frac{\frac{dv}{dx}}{v\sqrt{v^2-1}}.$$

### 73. Differentiation of arc csc $v$.[*]

Following method of last section,

XXIV
$$\frac{d}{dx}(\text{arc} \csc v) = -\frac{\frac{dv}{dx}}{v\sqrt{v_2-1}}.$$

### 74. Differentiation of arc vers $v$.

Let $\qquad y = \text{arc vers}\, v.;$[†]

then $\qquad v = \text{vers}\, y.$

[*] This function is defined for all values of $v$ except those lying between $-1$ and $+1$, and is seen to be many-valued. To make the function single-valued, $y$ is taken as the arc of smallest numerical value whose secant is $v$. This means that if $v$ is positive we confine ourselves to points on arc $AB$ (Fig. $a$), $y$ taking on values beween 0 and $\frac{\pi}{2}\left(\frac{\pi}{2}\right.$ may be included$\left.\right)$; and if $v$ is negative we confine ourselves to points on arc $CD$, $y$ taking on values between $-\pi$ and $-\frac{\pi}{2}$ $\left(-\frac{\pi}{2}\right.$ may be included$\left.\right)$.

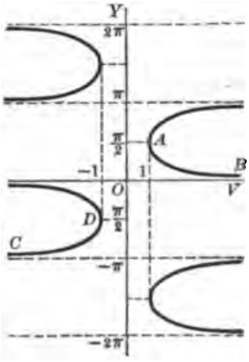


FIG. $a$

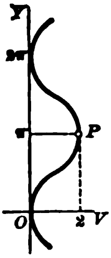


FIG. $b$

[†] Defined only for values of $v$ between 0 and 2 inclusive, and is many-valued. To make the function continuous, $y$ is taken as the smallest positive arc whose versed sine is $v$; that is, $y$ lies between 0 and $\pi$ inclusive. Hence we confine ourselves to arc $OP$ of the graph (Fig. $b$).

Differentiating with respect to $y$ by XVIII,

$$\frac{dv}{dy} = \sin y;$$

therefore

$$\frac{dy}{dv} = \frac{1}{\sin y}. \qquad \text{By } (C), \text{ p. } 58$$

But since $v$ is a function of $x$, this may be substituted in the formula

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}, \qquad (A), \text{ p. } 57$$

giving

$$\frac{dy}{dx} = \frac{1}{\sin y} \cdot \frac{dv}{dx}$$

$$= \frac{1}{\sqrt{2v - v^2}} \frac{dv}{dx}.$$

$\left[\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - (1 - \text{vers } y)^2} = \sqrt{2v - v^2}\right.$, the plus sign of the radical being taken since $\sin y$ is positive for all values of $y$ between 0 and $\pi$ inclusive.$\left.\right]$

$$\text{XXV} \qquad \therefore \quad \frac{d}{dx}(\text{arc vers } v) = \frac{\frac{dv}{dx}}{\sqrt{2v - v^2}}.$$

### EXAMPLES

Differentiate the following.

1. $y = \text{arc tan } ax^2$.

*Solution.*

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(ax^2)}{1 + (ax^2)^2} \qquad \text{by XXI}$$

$$[v = ax^2.]$$

$$= \frac{2ax}{1 + a^2x^4}.$$

2. $y = \text{arc sin}(3x - 4x^3)$.

*Solution.*

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(3x - 4x^3)}{\sqrt{1 - (3x - 4x^3)^2}} \qquad \text{by XIX}$$

$$[v = 3x - 4x^3.]$$

$$= \frac{3 - 12x^2}{\sqrt{1 - 9x^2 + 24x^4 - 16x^6}} = \frac{3}{\sqrt{1 - x^2}}.$$

3. $y = \text{arc sec}\,\frac{x^2+1}{x^2-1}.$

*Solution.*

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)}{\frac{x^2+1}{x^2-1}\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}} \qquad \text{by XXIII}$$

$$\left[v = \frac{x^2+1}{x^2-1}.\right]$$

$$= \frac{\frac{(x^2-1)\,2x-(x^2+1)\,2x}{(x^2-1)^2}}{\frac{x^2+1}{x^2-1}\cdot\frac{2x}{x^2-1}} = -\frac{2}{x^2+1}.$$

4. $y = \text{arc sin}\,\frac{x}{a}.$ $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}.$

5. $y = \text{arc cot}\,(x^2-5).$ $\qquad \frac{dy}{dx} = \frac{-2x}{1+(x^2-5)^2}.$

6. $y = \text{arc tan}\,\frac{2x}{1-x^2}.$ $\qquad \frac{dy}{dx} = \frac{2}{1+x^2}.$

7. $y = \text{arc cosec}\,\frac{1}{2x^2-1}.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}.$

8. $y = \text{arc vers}\,2x^2.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}.$

9. $y = \text{arc tan}\,\sqrt{1-x}.$ $\qquad \frac{dy}{dx} = -\frac{1}{2\sqrt{1-x}\,(2-x)}.$

10. $y = \text{arc cosec}\,\frac{3}{2x}.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{9-4x^2}}.$

11. $y = \text{arc vers}\,\frac{2x^2}{1+x^2}.$ $\qquad \frac{dy}{dx} = \frac{2}{1+x^2}.$

12. $y = \text{arc tan}\,\frac{x}{a}.$ $\qquad \frac{dy}{dx} = \frac{a}{a^2+x^2}.$

13. $y = \text{arc sin}\,\frac{x+1}{\sqrt{2}}.$ $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-2x-x^2}}.$

14. $f(x) = x\sqrt{a^2-x^2} + a^2\,\text{arc sin}\,\frac{x}{a}.$ $\qquad f'(x) = 2\sqrt{a^2-x^2}.$

15. $f(x) = \sqrt{a^2-x^2} + a\,\text{arc sin}\,\frac{x}{a}.$ $\qquad f'(x) = \left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}.$

16. $x = r\,\text{arc vers}\,\frac{y}{r} - \sqrt{2ry-y^2}.$ $\qquad \frac{dx}{dy} = \frac{y}{\sqrt{2ry-y^2}}.$

17. $\theta = \text{arc sin}\,(3r-1).$ $\qquad \frac{d\theta}{dr} = \frac{3}{\sqrt{6r-9r^2}}.$

18. $\phi = \operatorname{arc\,tan}\frac{r+a}{1-ar}.$ $\qquad \frac{d\phi}{dr} = \frac{1}{1+r^2}.$

19. $s = \operatorname{arc\,sec}\frac{1}{\sqrt{1-t^2}}.$ $\qquad \frac{ds}{dt} = \frac{1}{\sqrt{1-t^2}}.$

20. $\frac{d}{dx}(x \operatorname{arc\,sin} x) = \operatorname{arc\,sin} x + \frac{x}{\sqrt{1-x^2}}.$

21. $\frac{d}{d\theta}(\tan\theta \operatorname{arc\,tan}\theta) = \sec^2\theta \operatorname{arc\,tan}\theta + \frac{\tan\theta}{1+\theta^2}.$

22. $\frac{d}{dt}[\log(\operatorname{arc\,cos} t)] = -\frac{1}{\operatorname{arc\,cos} t\sqrt{1-t^2}}.$

23. $f(y) = \operatorname{arc\,cos}(\log y).$ $\qquad f'(y) = -\frac{1}{y\sqrt{1-(\log y)^2}}.$

24. $f(\theta) = \operatorname{arc\,sin}\sqrt{\sin\theta}.$ $\qquad f'(\theta) = \frac{1}{2}\sqrt{1+\operatorname{cosec}\theta}.$

25. $f(\phi) = \operatorname{arc\,tan}\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}.$ $\qquad f'(\phi) = \frac{1}{2}.$

26. $p = e^{\operatorname{arc\,tan} q}.$ $\qquad \frac{dp}{dq} = \frac{e^{\operatorname{arc\,tan} q}}{1+q^2}.$

27. $u = \operatorname{arc\,tan}\frac{e^v - e^{-v}}{2}.$ $\qquad \frac{du}{dv} = \frac{2}{e^v - e^{-v}}.$

28. $s = \operatorname{arc\,cos}\frac{e^t - e^{-t}}{e^t + e^{-t}}.$ $\qquad \frac{ds}{dt} = -\frac{2}{e^t + e^{-t}}.$

29. $y = x^{\operatorname{arc\,sin} x}.$ $\qquad \frac{dy}{dx} = x^{\operatorname{arc\,sin} x}\left(\frac{\operatorname{arc\,sin} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right).$

30. $y = e^{x^x}\operatorname{arc\,tan} x.$ $\qquad \frac{dy}{dx} = e^{x^x}\left[\frac{1}{1+x^2} + x^x \operatorname{arc\,tan} x\,(1+\log x)\right].$

31. $y = \operatorname{arc\,sin}(\sin x).$ $\qquad \frac{dy}{dx} = 1.$

32. $y = \operatorname{arc\,tan}\frac{4\sin x}{3+5\cos x}.$ $\qquad \frac{dy}{dx} = \frac{4}{5+3\cos x}.$

33. $y = \operatorname{arc\,cot}\frac{a}{x} + \log\sqrt{\frac{x-a}{x+a}}.$ $\qquad \frac{dy}{dx} = \frac{2ax^2}{x^4-a^4}.$

34. $y = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} - \frac{1}{2}\operatorname{arc\,tan} x.$ $\qquad \frac{dy}{dx} = \frac{x^2}{1-x^4}.$

35. $y = \sqrt{1-x^2}\operatorname{arc\,sin} x - x.$ $\qquad \frac{dy}{dx} = -\frac{x \operatorname{arc\,sin} x}{\sqrt{1-x^2}}.$

36. $y = \operatorname{arc\,cos}\frac{x^{2n}-1}{x^{2n}+1}.$ $\qquad \frac{dy}{dx} = -\frac{2nx^{n-1}}{x^{2n}+1}.$

Formulas (*A*), p. 57, for differentiating a *function of a function*, and (*C*), p. 58, for differentiating *inverse functions*, have been added to the list of formulas at the beginning of this chapter as XXVI and XXVII respectively.

In the next eight examples, first find $\frac{dy}{dv}$ and $\frac{dv}{dx}$ by differentiation and then substitute the results in

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx} \qquad \text{by XXVI}$$

to find $\frac{dy}{dx}$.[*]

In general our results should be expressed explicitly in terms of the independent variable; that is, $\frac{dy}{dx}$ in terms of $x$, $\frac{dx}{dy}$ in terms of $y$, $\frac{d\phi}{d\theta}$ in terms of $\theta$, etc.

37. $y = 2v^2 - 4$, $v = 3x^2 + 1$.

$\frac{dy}{dv} = 4v$; $\frac{dv}{dx} = 6x$; substituting in XXVI,

$$\frac{dy}{dx} = 4v\cdot 6x = 24x(3x^2+1).$$

38. $y = \tan 2v$, $v = \text{arc}\tan(2x-1)$.

$\frac{dy}{dv} = 2\sec^2 2v$; $\frac{dv}{dx} = \frac{1}{2x^2-2x+1}$; substituting in XXVI,

$$\frac{dy}{dx} = \frac{2\sec^2 2v}{2x^2-2x+1} = 2\,\frac{\tan^2 2v+1}{2x^2-2x+1} = \frac{2x^2-2x+1}{2(x-x^2)^2}.$$

$\left[\text{Since } v = \text{arc}\tan(2x-1),\ \tan v = 2x-1,\ \tan 2v = \frac{2x-1}{2x-2x^2}.\right]$

39. $y = 3v^2 - 4v + 5$, $v = 2x^3 - 5$. $\qquad \frac{dy}{dx} = 72x^5 - 204x^2$.

40. $y = \frac{2v}{3v-2}$, $v = \frac{x}{2x-1}$. $\qquad \frac{dy}{dx} = \frac{4}{(x-2)^2}$.

41. $y = \log(a^2 - v^2)$, $v = a\sin x$. $\qquad \frac{dy}{dx} = -2\tan x$.

42. $y = \text{arc}\tan(a + v)$, $v = e^x$. $\qquad \frac{dy}{dx} = \frac{e^x}{1+(a+e^x)^2}$.

43. $r = e^{2s} + e^s$, $s = \log(t - t^2)$. $\qquad \frac{dr}{dt} = 4t^3 - 6t^2 + 1$.

44. $w = \frac{1}{6}\log\frac{(v+1)^2}{v^2-v+1} - \frac{1}{\sqrt{3}}\text{arc}\tan\frac{2v-1}{\sqrt{3}}$,

$v = \frac{\sqrt[3]{1+3z+3z^2}}{z}$. $\qquad \frac{du}{dz} = \frac{1}{zv(1+z)}$.

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[*] As was pointed out on p. 57, it might be possible to eliminate $v$ between the two given expressions so as to find $y$ directly as a function of $x$, but in most cases the above method is to be preferred.

In the following examples first find $\frac{dx}{dy}$ by differentiation and then substitute in

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \text{XXVII}$$

to find $\frac{dy}{dx}$.

45. $x = y\sqrt{1+y}$. $\qquad \frac{dy}{dx} = \frac{2+3y}{2\sqrt{1+y}} = \frac{2y+3y^2}{2x}$.

46. $x = \sqrt{1+\cos y}$. $\qquad \frac{dy}{dx} = -\frac{2\sqrt{1+\cos y}}{\sin y} = -\frac{2}{\sqrt{2-x^2}}$.

47. $x = \frac{y}{1+\log y}$. $\qquad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y} = \frac{y^2}{xy - x^2}$.

48. $x = a\log\frac{a+\sqrt{a^2-y^2}}{y}$. $\qquad \frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$.

49. $x = r \operatorname{arc\,vers}\frac{y}{r} - \sqrt{2ry - y^2}$. $\qquad \frac{dy}{dx} = \sqrt{\frac{2r-y}{y}}$.

50. $r = \frac{s}{1+\log s}$. $\qquad \frac{dr}{ds} = \frac{r(s-r)}{s^2}$.

51. $u = \log\frac{e^v + \sqrt{e^{2v}-4}}{2}$. $\qquad \frac{dv}{du} = \frac{e^{2v}-1}{e^{2v}+1}$.

52. Show that the geometrical significance of XXVII is that the tangent makes complementary angles with the two coördinate axes.

**75. Implicit functions.** When a relation between $x$ and $y$ is given by means of an equation *not solved for* $y$, then $y$ is called an *implicit function* of $x$. For example, the equation

$$x^2 - 4y = 0$$

defines $y$ as an implicit function of $x$. Evidently $x$ is also defined by means of this equation as an implicit function of $y$. Similarly

$$x^2 + y^2 + z^2 - a^2 = 0$$

defines any one of the three variables as an implicit function of the other two.

It is sometimes possible to solve the equation defining an implicit function for one of the variables and thus change it into an explicit function. For instance, the above two implicit functions may be solved for $y$ giving

$$y = \frac{x^2}{4}$$

and

$$y = \pm\sqrt{x^2 + z^2};$$

the first showing $y$ as an explicit function of $x$, and the second as an explicit function of $x$ and $z$. In a given case, however, such a solution may be either impossible or too complicated for convenient use.

The two implicit functions used in this article for illustration may be respectively denoted by

$$f(x, y) = 0$$

and

$$F(x, y, z) = 0.$$

**76. Differentiation of implicit functions.** When $y$ is defined as an implicit function of $x$ by means of an equation in the form

$$(A) \qquad f(x, y) = 0,$$

it was explained in the last section how it might be inconvenient to solve for $y$ in terms of $x$; that is, to find $y$ as an explicit function of $x$ so that the formulas we have deduced in this chapter may be applied directly. Such, for instance, would be the case for the equation

$$(B) \qquad ax^6 + 2x^3y - y^7x - 10 = 0.$$

We then follow the rule:

*Differentiate, regarding y as a function of x, and put the result equal to zero.** That is,

$$(C) \qquad \frac{d}{dx}f(x, y) = 0.$$

Let us apply this rule in finding $\frac{dy}{dx}$ from $(B)$.

$$\frac{d}{dx}(ax^6 + 2x^3y - y^7x - 10) = 0; \qquad \text{by } (C)$$

$$\frac{d}{dx}(ax^6) + \frac{d}{dx}(2x^3y) - \frac{d}{dx}(y^7x) - \frac{d}{dx}(10) = 0;$$

$$6ax^5 + 2x^3\frac{dy}{dx} + 6x^2y - y^7 - 7xy^6\frac{dy}{dx} = 0;$$

$$(2x^3 - 7xy^6)\frac{dy}{dx} = y^7 - 6ax^5 - 6x^2y;$$

$$\frac{dy}{dx} = \frac{y^7 - 6ax^5 - 6x^2y}{2x^3 - 7xy^6}. \qquad \textit{Ans.}$$

The student should observe that in general the result will contain both $x$ and $y$.

---

* This process will be justified in § 136, p. 260. Only corresponding values of $x$ and $y$ which satisfy the given equation may be substituted in the derivative.

### EXAMPLES

Differentiate the following by the above rule.

1. $y^2 = 4px.$ $\qquad \frac{dy}{dx} = \frac{2p}{y}.$

2. $x^2 + y^2 = r^2.$ $\qquad \frac{dy}{dx} = -\frac{x}{y}.$

3. $b^2x^2 + a^2y^2 = a^2b^2.$ $\qquad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$

4. $y^3 - 3y + 2ax = 0.$ $\qquad \frac{dy}{dx} = \frac{2a}{3(1 - y^2)}.$

5. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$ $\qquad \frac{dy}{dx} = -\sqrt{\frac{y}{x}}.$

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$ $\qquad \frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}.$

7. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$ $\qquad \frac{dy}{dx} = -\frac{3b^{\frac{2}{3}}xy^{\frac{1}{3}}}{a^2}.$

8. $y^2 - 2xy + b^2 = 0.$ $\qquad \frac{dy}{dx} = \frac{y}{y - x}.$

9. $x^3 + y^3 - 3axy = 0.$ $\qquad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$

10. $x^y = y^x.$ $\qquad \frac{dy}{dx} = \frac{y^2 - xy\log y}{x^2 - xy\log x}.$

11. $\rho^2 = a^2\cos 2\theta.$ $\qquad \frac{d\rho}{d\theta} = -\frac{a^2\sin 2\theta}{\rho}.$

12. $\rho^2\cos\theta = a^2\sin 3\theta.$ $\qquad \frac{d\rho}{d\theta} = \frac{3a^2\cos 3\theta + \rho^2\sin\theta}{2\rho\cos\theta}.$

13. $\cos(uv) = cv.$ $\qquad \frac{du}{dv} = \frac{c + u\sin(uv)}{v\sin(uv)}.$

14. $\theta = \cos(\theta + \phi).$ $\qquad \frac{d\theta}{d\phi} = -\frac{\sin(\theta + \phi)}{1 + \sin(\theta + \phi)}.$

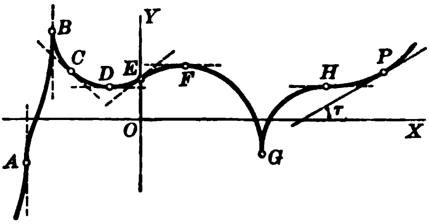
# CHAPTER VII

## SIMPLE APPLICATIONS OF THE DERIVATIVE

**77. Direction of a curve.** It was shown in § 45, p. 44, that **if**

$$y = f(x)$$

is the equation of a curve (see figure), then



$$\frac{dy}{dx} = \tan\tau = \textbf{slope of line tangent to curve at any point } P.$$

The *direction of a curve* at any point is defined to be the same as the direction of the line tangent to the curve at that point. From this it follows at once that

$$\frac{dy}{dx} = \tan\tau = \textbf{slope of curve at any point } P.$$

At a particular point whose coördinates are known we write

$$\left[\frac{dy}{dx}\right]_{\substack{x=x_1\\y=y_1}} = \textbf{slope of curve (or tangent) at point } (x_1, y_1).$$

At points such as $D$, $F$, $H$, where the curve (or tangent) is *parallel to the axis of* $X$,

$$\tau = 0°, \textbf{ therefore } \frac{dy}{dx} = 0.$$

At points such as $A$, $B$, $G$, where the curve (or tangent) is *perpendicular to the axis of* $X$,

$$\tau = 90°, \textbf{ therefore } \frac{dy}{dx} = \infty.$$

At points such as $E$, where the *curve is rising*,*

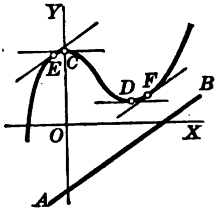
**$\tau$ = an acute angle, therefore $\frac{dy}{dx}$ = a positive number.**

The curve (or tangent) has a positive slope to the left of $B$, between $D$ and $F$, and to the right of $G$.

At points such as $C$, where the *curve is falling*,*

**$\tau$ = an obtuse angle, therefore $\frac{dy}{dx}$ = a negative number.**

The curve (or tangent) has a negative slope between $B$ and $D$ and between $F$ and $G$.



Ex. 1. Given the curve $y = \frac{x^3}{3} - x^2 + 2$ (see figure).

(a) Find $\tau$ when $x = 1$.

(b) Find $\tau$ when $x = 3$.

(c) Find the points where the curve is parallel to $OX$.

(d) Find the points where $\tau = 45°$.

(e) Find the points where the curve is parallel to the line $2x - 3y = 6$ (line $AB$).

*Solution.* Differentiating, $\frac{dy}{dx} = x^2 - 2x$ = slope at any point.

(a) $\tan\tau = \left[\frac{dy}{dx}\right]_{x=1} = 1 - 2 = -1$; therefore $\tau = 135°$. *Ans.*

(b) $\tan\tau = \left[\frac{dy}{dx}\right]_{x=3} = 9 - 6 = 3$; therefore $\tau = \text{arc}\tan 3$. *Ans.*

(c) $\tau = 0°$, $\tan\tau = \frac{dy}{dx} = 0$; therefore $x^2 - 2x = 0$. Solving this equation, we find that $x = 0$ or $2$, giving points $C$ and $D$ where curve (or tangent) is parallel to $OX$.

(d) $\tau = 45°$, $\tan\tau = \frac{dy}{dx} = 1$; therefore $x^2 - 2x = 1$. Solving, we get $x = 1 \pm \sqrt{2}$, giving two points where the slope of curve (or tangent) is unity.

(e) Slope of line $= \frac{2}{3}$; therefore $x^2 - 2x = \frac{2}{3}$. Solving, we get $x = 1 \pm \sqrt{\frac{5}{3}}$, giving points $E$ and $F$ where curve (or tangent) is parallel to line $AB$.

Since a curve at any point has the same direction as its tangent at that point, the angle between two curves at a common point will be the angle between their tangents at that point.

Ex. 2. Find the angle of intersection of the circles

$(A)$ $\quad x^2 + y^2 - 4x = 1,$

$(B)$ $\quad x^2 + y^2 - 2y = 9.$

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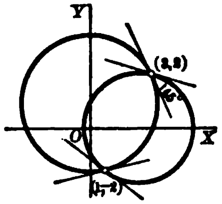
* When moving from left to right on curve.

*Solution.* Solving simultaneously we find the points of intersection to be (3, 2) and (1, − 2).



$$\frac{dy}{dx} = \frac{2 - x}{y} \text{ from } (A). \qquad \text{By § 76, p. 84}$$

$$\frac{dy}{dx} = \frac{x}{1 - y} \text{ from } (B). \qquad \text{By § 76, p. 84}$$

$$\left[\frac{2 - x}{y}\right]_{\substack{x=3\\y=2}} = -\tfrac{1}{2} = \text{slope of tangent to } (A) \text{ at } (3, 2).$$

$$\left[\frac{x}{1 - y}\right]_{\substack{x=3\\y=2}} = -3 = \text{slope of tangent to } (B) \text{ at } (3, 2).$$

The formula for finding the angle between two lines whose slopes are $m_1$ and $m_2$ is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}. \qquad 55, \text{ p. } 3$$

Substituting, $\tan\theta = \dfrac{-\frac{1}{2} + 3}{1 + \frac{3}{2}} = 1$; therefore $\theta = 45°$. *Ans.*

This is also the angle of intersection at point (1, − 2).

## EXAMPLES

The corresponding figure should be drawn in each of the following examples.

1. Find the slope of $y = \dfrac{x}{1 + x^2}$ at the origin. *Ans.* $1 = \tan\tau$.

2. What angle does the tangent to the curve $x^2y^2 = a^3(x + y)$ at the origin make with the axis of $X$? *Ans.* $\tau = 135°$.

3. What is the direction in which the point generating the graph of $y = 3x^2 - x$ tends to move at the instant when $x = 1$?
*Ans.* Parallel to a line whose slope is 5.

4. Show that $\dfrac{dy}{dx}$ (or slope) is constant for a straight line.

5. Find the points where the curve $y = x^3 - 3x^2 - 9x + 5$ is parallel to the axis of $X$. *Ans.* $x = 3$ and $x = 1$.

6. Find the points where the curve $y(x - 1)(x - 2) = x - 3$ is parallel to the axis of $X$. *Ans.* $x = 3 \pm \sqrt{2}$.

7. At what point on $y^2 = 2x^3$ is the slope equal to 3? *Ans.* (2, 4).

8. At what points on the circle $x^2 + y^2 = r^2$ is the slope of tangent line equal to $-\frac{3}{4}$? *Ans.* $\left(\pm\dfrac{3r}{5}, \pm\dfrac{4r}{5}\right)$.

9. Where is the tangent to the parabola $y = x^2 - 7x + 3$ parallel to the line $y = 5x + 2$? *Ans.* (6, − 3).

10. Find the points where the tangent to the circle $x^2 + y^2 = 169$ is perpendicular to the line $5x + 12y = 60$. *Ans.* (± 12, ± 5).

**11.** Find the point where the tangent to the parabola $y^2 = 4ax$ is parallel to the circle $x^2 + y^2 = 2rx$. *Ans.* $x = r - 2a$.

**12.** At what angles does the line $3y - 2x - 8 = 0$ cut the parabola $y^2 = 8x$. *Ans.* arc tan $\frac{1}{5}$, and arc tan $\frac{1}{8}$.

**13.** Find the angle of intersection between the parabola $y^2 = 6x$ and the circle $x^2 + y^2 = 16$. *Ans.* arc tan $\frac{5}{3}\sqrt{3}$.

**14.** Show that the hyperbola $x^2 - y^2 = 5$ and the ellipse $\frac{x^2}{18} + \frac{y^2}{8} = 1$ intersect at right angles.

**15.** Show that the circle $x^2 + y^2 = 8ax$ and the cissoid $y^2 = \frac{x^3}{2a - x}$

(a) are perpendicular at the origin;

(b) intersect at an angle of 45° at two other points.

**16.** Find the angle of intersection of the parabola $x^2 = 4ay$ and the witch $y = \frac{8a^3}{x^2 + 4a^2}$. *Ans.* arc tan $3 = 71° 33'.9$.

**17.** Show that the tangents to the folium of Descartes $x^3 + y^3 = 3axy$ at the points where it meets the parabola $y^2 = ax$ are parallel to the axis of $Y$.

**18.** At how many points can the curve $y = x^3 - 2x^2 + x - 4$ be parallel to the axis of $X$? What are the points? *Ans.* Two; at $(1, -4)$ and $(\frac{1}{3}, -\frac{104}{27})$.

**19.** Find the angle at which the parabolas $y = 3x^2 - 1$ and $y = 2x^2 + 3$ intersect. *Ans.* arc tan $\frac{4}{97}$.

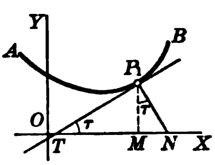
**20.** Find the relation between the coefficients of the conics $a_1x^2 + b_1y^2 = 1$ and $a_2x^2 + b_2y^2 = 1$ when they intersect at right angles. *Ans.* $\frac{1}{a_1} - \frac{1}{b_1} = \frac{1}{a_2} - \frac{1}{b_2}$.

**78. Equations of tangent and normal, lengths of subtangent and subnormal. Rectangular coördinates.** The equation of a straight line passing through the point $(x_1, y_1)$ and having the slope $m$ is

$$y - y_1 = m(x - x_1). \qquad \text{54 (c), p. 3}$$

If this line is tangent to the curve $AB$ at the point $P_1(x_1, y_1)$, then from § 77, p. 86,

$$m = \tan\tau = \left[\frac{dy}{dx}\right]_{\substack{x=x_1\\y=y_1}} = \frac{dy_1}{dx_1}.^*$$



Hence at point of contact $P_1(x_1, y_1)$ the *equation of the tangent line* $TP_1$ is

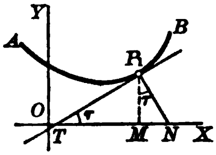
$$(1) \qquad y - y_1 = \frac{dy_1}{dx_1}(x - x_1).$$

---

[*] By this notation is meant that we should first find $\frac{dy}{dx}$, then in the result substitute $x_1$ for $x$ and $y_1$ for $y$. The student is warned against interpreting the symbol $\frac{dy_1}{dx_1}$ to mean *the derivative of* $y_1$ *with respect to* $x_1$, for that has no meaning whatever since $x_1$ and $y_1$ are both constants.

The normal being perpendicular to tangent, its slope is

$$-\frac{1}{m}=-\frac{dx_1}{dy_1}. \qquad \text{By 55, p. 3}$$



And since it also passes through the point of contact $P_1(x_1, y_1)$, we have for the *equation of the normal* $P_1N$

$$y-y_1=-\frac{dx_1}{dy_1}(x-x_1). \tag{2}$$

That portion of the tangent which is intercepted between the point of contact and $OX$ is called the *length of the tangent* $(=TP_1)$, and its projection on the axis of $X$ is called the *length of the subtangent* $(=TM)$. Similarly we have the *length of the normal* $(=P_1N)$ and the *length of the subnormal* $(=MN)$.

In triangle $TP_1M$, $\tan\tau=\frac{MP_1}{TM}$; therefore

$$TM^*=\frac{MP_1}{\tan\tau}=y_1\frac{dx_1}{dy_1}=\textbf{length of subtangent.} \tag{3}$$

In the triangle $MP_1N$, $\tan\tau=\frac{MN}{MP_1}$; therefore

$$MN^\dagger=MP_1\tan\tau=y_1\frac{dy_1}{dx_1}=\textbf{length of subnormal.} \tag{4}$$

The length of tangent $(=TP_1)$ and the length of normal $(=P_1N)$ may then be found directly from the figure, each being the hypotenuse of a right triangle having the two legs known. Thus

$$TP_1=\sqrt{\overline{TM}^2+\overline{MP_1}^2}=\sqrt{\left(y_1\frac{dx_1}{dy_1}\right)^2+(y_1)^2}$$

$$=y_1\sqrt{\left(\frac{dx_1}{dy_1}\right)^2+1}=\textbf{length of tangent.} \tag{5}$$

$$P_1N=\sqrt{\overline{MP_1}^2+\overline{MN}^2}=\sqrt{(y_1)^2+\left(y_1\frac{dy_1}{dx_1}\right)^2}$$

$$=y_1\sqrt{1+\left(\frac{dy_1}{dx_1}\right)^2}=\textbf{length of normal.} \tag{6}$$

The student is advised to get the lengths of the tangent and of the normal directly from the figure rather than by using (5) and (6) as formulas.

---

[*] If subtangent extends to the right of $T$, we consider it positive; if to the left, negative.

[†] If subnormal extends to the right of $M$, we consider it positive; if to the left, negative.

## EXAMPLES

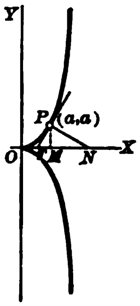
**1. Find the equations of tangent and normal, lengths of subtangent, subnormal tangent, and normal at the point $(a, a)$ on the cissoid $y^2 = \frac{x^3}{2a - x}$.**

*Solution.*

$$\frac{dy}{dx} = \frac{3ax^2 - x^3}{y(2a - x)^2}.$$

Hence

$$\frac{dy_1}{dx_1} = \left[\frac{dy}{dx}\right]_{\substack{x=a\\y=a}} = \frac{3a^3 - a^3}{a(2a - a)^2} = 2 = \text{slope}.$$



Substituting in (1) gives

$$y = 2x - a, \text{ equation of tangent.}$$

Substituting in (2) gives

$$2y + x = 3a, \text{ equation of normal.}$$

Substituting in (3) gives

$$TM = \frac{a}{2} = \text{length of subtangent.}$$

Substituting in (4) gives

$$MN = 2a = \text{length of subnormal.}$$

Also,

$$PT = \sqrt{(TM)^2 + (MP)^2} = \sqrt{\frac{a^2}{4} + a^2} = \frac{a}{2}\sqrt{5} = \text{length of tangent,}$$

and

$$PN = \sqrt{(MN)^2 + (MP)^2} = \sqrt{4a^2 + a^2} = a\sqrt{5} = \text{length of normal.}$$

**2. Find equations of tangent and normal to the ellipse $x^2 + 2y^2 - 2xy - x = 0$ at the points where $x = 1$.**

*Ans.* At $(1, 0)$, $2y = x - 1$, $y + 2x = 2$.
At $(1, 1)$, $2y = x + 1$, $y + 2x = 3$.

**3. Find equations of the tangent and normal, lengths of subtangent and subnormal at the point $(x_1, y_1)$ on the circle $x^2 + y^2 = r^2$.**

*Ans.* $x_1x + y_1y = r^2$, $x_1y - y_1x = 0$, $-\frac{y_1^2}{x_1}$, $x_1$.

**4. Show that the subtangent to the parabola $y^2 = 4px$ is bisected at the vertex, and that the subnormal is constant and equal to $2p$.**

**5. Find the equation of tangent at $(x_1, y_1)$**

(a) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; (b) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

*Ans.* (a) $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$; (b) $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$.

**6. Find equations of tangent and normal to the witch $y = \frac{8a^3}{4a^2 + x^2}$ at the point where $x = 2a$.**

*Ans.* $x + 2y = 4a$, $y = 2x - 3a$.

7. Prove that at any point on the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ the lengths of subnormal and normal are $\frac{a}{4}\left(e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}\right)$ and $\frac{y^2}{a}$ respectively.

8. Find the equation of tangent to the conic $ax^2 + 2bxy + cy^2 + 2dx + 2cy + f = 0$ at the point $(x_1, y_1)$.

*Ans.* $ax_1x + b(y_1x + x_1y) + cy_1y + d(x_1 + x) + e(y_1 + y) + f = 0$.*

9. Show that the equation of tangent to curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at the point $(a, b)$ is $\frac{x}{a} + \frac{y}{b} = 2$ for all values of $n$.

10. Prove that the length of subtangent to $y = a^x$ is constant and equal to $\frac{1}{\log a}$.

11. Get the equation of tangent to the parabola $y^2 = 20x$ which makes an angle of 45° with the axis of $x$. *Ans.* $y = x + 5$.

*Hint.* First find point of contact by method of Ex. 1 (e), p. 87.

12. Find equations of tangents to the circle $x^2 + y^2 = 52$ which are parallel to the line $2x + 3y = 6$. *Ans.* $2x + 3y \pm 26 = 0$.

13. Find equations of tangents to the hyperbola $4x^2 - 9y^2 + 36 = 0$ which are perpendicular to the line $2y + 5x = 10$. *Ans.* $2x - 5y \pm 8 = 0$.

14. Show that in the equilateral hyperbola $2xy = a^2$ the area of the triangle formed by a tangent and the coördinate axes is constant and equal to $a^2$.

15. Find equations of tangents and normals to the curve $y^2 = 2x^2 - x^3$ at the points where $x = 1$. *Ans.* At $(1, 1)$, $2y = x + 1$, $y + 2x = 3$.
At $(1, -1)$, $2y = -x - 1$, $y - 2x = -3$.

16. Show that the sum of the intercepts of the tangent to the parabola

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$$

on the coördinate axes is constant and equal to $a$.

17. Find the equation of tangent to the curve $x^2(x + y) = a^2(x - y)$ at the origin.
*Ans.* $y = x$.

18. Show that for the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ that portion of the tangent included between the coördinate axes is constant and equal to $a$.

19. Show that the curve $y = ae^{\frac{x}{c}}$ has a constant subtangent.

20. Show that the length of tangent is constant in the tractrix

$$x = \sqrt{c^2 - y^2} + \frac{c}{2}\log\frac{c - \sqrt{c^2 - y^2}}{c + \sqrt{c^2 - y^2}}.$$

**79. Parametric equations of a curve.** Let the equation of a curve be

$$(A) \qquad F(x, y) = 0.$$

If $x$ is given as a function of a third variable, $a$ say, called a *parameter*, then by virtue of $(A)$ $y$ is also a function of $a$, and the same functional relation (*A*) between $x$ and $y$ may generally be expressed by means of equations in the form

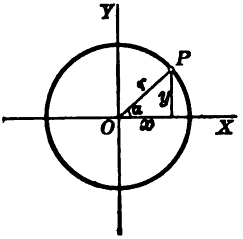
---

* In Exs. 3, 5, and 8 the student should notice that if we drop the subscripts in equations of tangents they reduce to the equations of the curves themselves.

$$(B)\qquad \begin{cases} x = f(a), \\ y = \phi(a); \end{cases}$$

each value of $a$ giving a value of $x$ and a value of $y$. Equations (*B*) are called *parametric equations* of the curve. If we eliminate $a$ between equations (*B*), it is evident that the relation (*A*) must result. For example, take equation of circle

$$x^2 + y^2 = r^2, \text{ or } y = \sqrt{r^2 - x^2}.$$



Let $x = r \cos a$; then

$y = r \sin a$, and we have

$$(C)\qquad \begin{cases} x = r \cos a, \\ y = r \sin a, \end{cases}$$

as parametric equations of the circle in figure, $a$ being the parameter.

If we eliminate $a$ between equations (*C*) by squaring and adding the results, we have

$$x^2 + y^2 = r^2(\cos^2 a + \sin^2 a) = r^2,$$

the rectangular equation of the circle. It is evident that if $a$ varies from 0 to $2\pi$, the point $P(x, y)$ will describe a complete circumference.

In § 84, p. 104, we shall discuss the motion of a point $P$, which motion is defined by equations such as

$$\begin{cases} x = f(t), \\ y = \phi(t). \end{cases}$$

We call these the parametric equations of the path, the time $t$ being the parameter. Thus in Ex. 2, p. 106, we see that

$$\begin{cases} x = v_0 \cos a \cdot t, \\ y = -\frac{1}{2} gt^2 + v_0 \sin a \cdot t \end{cases}$$

are really the parametric equations of the trajectory of a projectile, the time $t$ being the parameter. The elimination of $t$ gives the rectangular equation of the trajectory

$$y = x \tan a - \frac{gx^2}{2 v_0^2 \cos^2 a}.$$

Since from $(B)$ $y$ is given as a function of $a$, and $a$ as a function of $x$, we have

$$\frac{dy}{dx}=\frac{dy}{da}\cdot\frac{da}{dx} \qquad \text{by XXVI}$$

$$=\frac{dy}{da}\cdot\frac{1}{\dfrac{dx}{da}}; \qquad \text{by XXVII}$$

that is,

$$(D) \qquad \frac{dy}{dx}=\frac{\dfrac{dy}{da}}{\dfrac{dx}{da}}.$$

Hence, if parametric equations of a curve are given, we can find equations of tangent and normal, lengths of subtangent and subnormal at a given point on the curve, by first finding the value of $\frac{dy}{dx}$ at that point from $(D)$ and then substituting in formulas (1), (2), (3), (4) of last section.

**Ex. 1.** Find equations of tangent and normal, lengths of subtangent and subnormal to the ellipse

$$(E) \qquad \begin{cases} x = a\cos\phi, \\ y = b\sin\phi,^{*} \end{cases}$$

at the point where $\phi = \frac{\pi}{4}$

*Solution.* The parameter being $\phi$, $\frac{dx}{d\phi} = -a\sin\phi$, $\frac{dy}{d\phi} = b\cos\phi$.

Substituting in (D), $\frac{dy}{dx} = -\frac{b\cos\phi}{a\sin\phi}$ = slope at any point.

---

[*] As in figure draw the major and minor auxiliary circles of the ellipse. Through two points $B$ and $C$ on the same radius draw lines parallel to the axes of coördinates. These lines will intersect in a point $P(x, y)$ on the ellipse, because

$$x = OA = OB\cos\phi = a\cos\phi$$

and

$$y = AP = OD = OC\sin\phi = b\sin\phi,$$

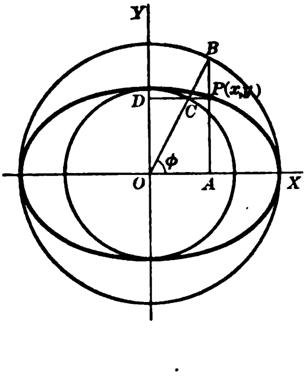
or,

$$\frac{x}{a} = \cos\phi \text{ and } \frac{y}{b} = \sin\phi.$$

Now squaring and adding we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi + \sin^2\phi = 1,$$

the rectangular equation of the ellipse. $\phi$ is sometimes called the eccentric angle of the ellipse.



Substituting $\phi = \frac{\pi}{4}$ in the given equations $(E)$, we get $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ as the point of contact. Hence

$$\frac{dy_1}{dx_1} = -\frac{b}{a}.$$

Substituting in (1), p. 89,

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{b}{\sqrt{2}}\right),$$

or, $bx + ay = \sqrt{2}\,ab$, equation of tangent.

Substituting in (2), p. 90,

$$y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right),$$

or, $ax - by = (a^2 - b^2)\sqrt{2}$, equation of normal.

Substituting in (3) and (4), p. 90,

$$\frac{bx}{\sqrt{2}}\left(-\frac{a}{b}\right) = -\frac{b^2}{a\sqrt{2}} = \text{length of subtangent.}$$

$$\frac{bx}{\sqrt{2}}\left(-\frac{a}{b}\right) = -\frac{a}{\sqrt{2}} = \text{length of subnormal.}$$

**Ex. 2.** Given equation of the cycloid* in parametric form

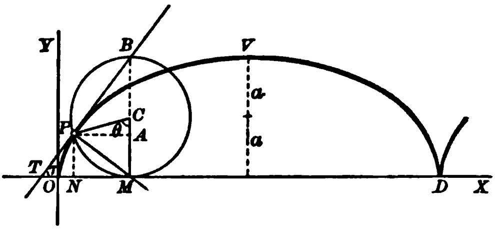
$$\begin{cases} x = a(\theta - \sin\theta), \\ y = b(1 - \cos\theta); \end{cases}$$

$\theta$ being the variable parameter. Find lengths of subtangent, subnormal, tangent, and normal at the point where $\theta = \frac{\pi}{2}$.

*Solution.* $\frac{dx}{d\theta} = a(1 - \cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$.

---

* The path described by a point on the circumference of a circle which rolls without sliding on a fixed straight line is called the cycloid. Let the radius of rolling circle be $a$, $P$ the generating point, and $M$ the point of contact with the fixed line $OX$, which is called the base. If arc



$PM$ equals $OM$ in length, then $P$ will touch at $O$ if circle is rolled to the left. We have, denoting angle $PCM$ by $\theta$,

$$x = OM - NM = a\theta - a\sin\theta = a(\theta - \sin\theta),$$
$$y = PN = MC - AC = a - a\cos\theta = a(1 - \cos\theta);$$

the parametric equations of the cycloid, the angle $\theta$ through which the rolling circle turns being the parameter. $OD = 2\pi a$ is called the base of one arch of the cycloid, and the point $V$ is called the vertex. Eliminating $\theta$, we get the rectangular equation

$$x = a \operatorname{arc} \cos\left(\frac{a - y}{a}\right) - \sqrt{2ay - y^2}.$$

Substituting in (*D*), p. 94, $\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}$ = slope at any point.

Since $\theta = \frac{\pi}{2}$, the point of contact is $\left(\frac{\pi a}{2} - a,\ a\right)$, and $\frac{dy_1}{dx_1} = 1$.

Substituting in (3), (4), (5), (6) of last section, we get

$$\begin{aligned} &\text{length of subtangent} = a,\\ &\text{length of subnormal} = a,\\ &\text{length of tangent} = a\sqrt{2},\\ &\text{length of normal} = a\sqrt{2}. \quad \textit{Ans.}\end{aligned}$$

NOTE. Draw the tangent *PT*, the vertical diameter *MB*, and connect *P* and *B*.

$$\tan MTP = \frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$

[From 37, p. 2, and 39, p. 3.]

Hence $$\text{angle } MTP = \frac{\pi}{2} - \frac{\theta}{2}. \qquad \text{By 29, p. 2}$$

Also, angle $PBM = \frac{\theta}{2}$, since it is measured by one half the arc *MP* which measures the central angle $\theta$, and we have

$$\text{angle } APB = \frac{\pi}{2} - \frac{\theta}{2}.$$

Comparing, we see that

$$\text{angle } MTP = \text{angle } APB.$$

Therefore:

*The tangent to a cycloid always passes through the highest point of the generating circle.*

## EXAMPLES

In the following curves find lengths of (a) subtangent, (b) subnormal, (c) tangent, (d) normal, at any point.

1. The curve $\begin{cases} x = a(\cos t + t\sin t),\\ y = a(\sin t - t\cos t).\end{cases}$

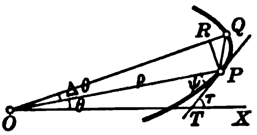
*Ans.* (a) $y\cot t$, (b) $y\tan t$, (c) $\frac{y}{\sin t}$, (d) $\frac{y}{\cos t}$.

2. The hypocycloid (astroid) $\begin{cases} x = 4\,a\cos^3 t, \\ y = 4\,a\sin^3 t. \end{cases}$

*Ans.* (a) $-y\cot t$, (b) $-y\tan t$, (c) $\dfrac{y}{\sin t}$, (d) $-\dfrac{y}{\cos t}$.

3. The cardioid $\begin{cases} x = a(2\cos t - \cos 2\,t), \\ y = a(2\sin t - \sin 2\,t). \end{cases}$

**80. Angle between the radius vector drawn to a point on a curve and the tangent to the curve at that point.** Let the equation of curve in polar coördinates be $\rho = f(\theta)$.



Let $P$ be any fixed point $(\rho, \theta)$ on the curve. If $\theta$, which we assume as the independent variable, takes on an increment $\Delta\theta$, then $\rho$ will take on a corresponding increment $\Delta\rho$. Denote by $Q$ the point $(\rho + \Delta\rho,\ \theta + \Delta\theta)$. Draw $PR$ perpendicular to $OQ$. Then $OQ = \rho + \Delta\rho$, $PR = \rho\sin\Delta\theta$, and $OR = \rho\cos\Delta\theta$. Also,

$$\tan PQR = \frac{PR}{RQ} = \frac{PR}{OQ - OR} = \frac{\rho\sin\Delta\theta}{\rho + \Delta\rho - \rho\cos\Delta\theta}.$$

Denote by $\psi$ the angle between the radius vector $OP$ and the tangent $PT$. If we now let $\Delta\theta$ approach the limit zero, then

(*a*) *the point $Q$ will approach indefinitely near $P$;*

(*b*) *the secant $PQ$ will approach the tangent $PT$ as a limiting position; and*

(*c*) *the angle $PQR$ will approach $\psi$ as a limit.*

Hence

$$\tan\psi = \underset{\Delta\theta = 0}{\text{limit}}\ \frac{\rho\sin\Delta\theta}{\rho + \Delta\rho - \rho\cos\theta}$$

$$= \underset{\Delta\theta = 0}{\text{limit}}\ \frac{\rho\sin\Delta\theta}{2\,\rho\sin^2\frac{\theta}{2} + \Delta\rho}$$

[Since from 39, p. 3, $\rho - \rho\cos\theta = \rho(1 - \cos\theta) = 2\rho\sin^2\frac{\theta}{2}$.]

$$= \underset{\Delta\theta = 0}{\text{limit}}\ \frac{\dfrac{\rho\sin\Delta\theta}{\Delta\theta}}{\dfrac{2\,\rho\sin^2\frac{\theta}{2}}{\Delta\theta} + \dfrac{\Delta\rho}{\Delta\theta}}$$

[Dividing both numerator and denominator by $\Delta t$.]

$$= \underset{\Delta\theta = 0}{\text{limit}} \frac{\rho \cdot \dfrac{\sin \Delta\theta}{\Delta\theta}}{\rho \sin \dfrac{\Delta\theta}{2} \cdot \dfrac{\sin \dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}} + \dfrac{\Delta\rho}{\Delta\theta}}.$$

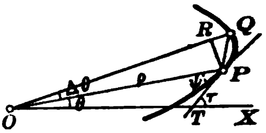
Since $\underset{\Delta\theta = 0}{\text{limit}} \left(\dfrac{\Delta\rho}{\Delta\theta}\right) = \dfrac{d\rho}{d\theta}$ and $\underset{\Delta\theta = 0}{\text{limit}} \left(\sin \dfrac{\Delta\theta}{2}\right) = 0$, also

$\underset{\Delta\theta = 0}{\text{limit}} \left(\dfrac{\sin \Delta\theta}{\Delta\theta}\right) = 1$ and $\underset{\Delta\theta = 0}{\text{limit}} \dfrac{\sin \dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}} = 1$ by (14), p. 30, we have

(*A*) $$\tan \psi = \frac{\rho}{\dfrac{d\rho}{d\theta}}.$$

From the triangle *OPT* we get

(*B*) $$\tau = \theta + \psi.$$



Ex. 1. Find $\psi$ and $\tau$ in the cardioid $\rho = a(1 - \cos\theta)$.

*Solution.* $\dfrac{d\rho}{d\theta} = a \sin\theta$. Substituting in (*A*) gives

$$\tan\psi = \frac{\rho}{\dfrac{d\rho}{d\theta}} = \frac{a(1 - \cos\theta)}{a \sin\theta} = \frac{2a \sin^2 \dfrac{\theta}{2}}{2a \sin\dfrac{\theta}{2} \cos\dfrac{\theta}{2}} = \tan\frac{\theta}{2}.$$ By 39, p. 3, and 37, p. 2

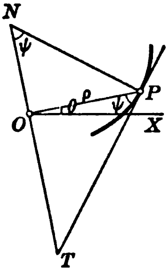
Since $\tan\psi = \tan\dfrac{\theta}{2}$, $\psi = \dfrac{\theta}{2}$. *Ans.* Substituting in (*B*), $\tau = \theta + \dfrac{\theta}{2} = \dfrac{3\theta}{2}$. *Ans.*

**81. Lengths of polar subtangent and polar subnormal.** Draw a line *NT* through the origin perpendicular to the radius vector of the point *P* on the curve. If *PT* is the tangent and *PN* the normal to the curve at *P*, then

$$OT = \textit{length of polar subtangent},$$

and $$ON = \textit{length of polar subnormal}$$

of the curve at *P*.



In the triangle $OPT$, $\tan\psi = \frac{OT}{\rho}\cdot$ Therefore

$$(7)\qquad OT = \rho\tan\psi = \rho^2\frac{d\theta}{d\rho} = \textbf{length of polar subtangent.*}$$

In the triangle $OPN$, $\tan\psi = \frac{\rho}{ON}\cdot$ Therefore

$$(8)\qquad ON = \frac{\rho}{\tan\psi} = \frac{d\rho}{d\theta} = \textbf{length of polar subnormal.}$$

The *length of the polar tangent* ($= PT$) and the *length of the polar normal* ($= PN$) may be found from the figure, each being the hypotenuse of a right triangle.

**Ex. 1.** Find lengths of polar subtangent and subnormal to the lemniscate $\rho^2 = a^2\cos 2\theta$.

*Solution.* Differentiating as an implicit function with respect to $\theta$,

$$2\rho\frac{d\rho}{d\theta} = -2a^2\sin 2\theta, \text{ or } \frac{d\rho}{d\theta} = -\frac{a^2\sin 2\theta}{\rho}.$$

Substituting in (7) and (8), we get

$$\text{length of polar subtangent} = -\frac{\rho^3}{a^2\sin 2\theta},$$

$$\text{length of polar subnormal} = -\frac{a^2\sin 2\theta}{\rho}.$$

If we wish to express the results in terms of $\theta$, find $\rho$ in terms of $\theta$ from the given equation and substitute. Thus, in above, $\rho = \pm a\sqrt{\cos 2\theta}$; therefore length of polar subtangent $= \pm a\cot 2\theta\sqrt{\cos 2\theta}$.

## EXAMPLES

1. In the circle $\rho = r\sin\theta$, find $\psi$ and $\tau$ in terms of $\theta$. *Ans.* $\psi = \theta$, $\tau = 2\theta$.

2. In the parabola $\rho = a\sec^2\frac{\theta}{2}$, show that $\tau + \psi = \pi$.

3. Show that $\psi$ is constant in the logarithmic spiral $\rho = e^{a\theta}$. Since the tangent makes a constant angle with the radius vector this curve is also called the equiangular spiral.

4. Given the concoid $\rho = a\sin^3\frac{\theta}{3}$; prove that $\tau = 4\psi$.

5. Show that $\tan\psi = \theta$ in the spiral of Archimedes $\rho = a\theta$. Find values of $\psi$ when $\theta = 2\pi$ and $4\pi$. *Ans.* $\psi = 80°\,57'$ and $85°\,27'$.

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[*] When $\theta$ increases with $\rho$, $\frac{d\theta}{d\rho}$ is positive and $\psi$ is an acute angle, as in above figure. Then the subtangent $OT$ is positive and is measured to the right of an observer placed at $O$ and looking along $OP$. When $\frac{d\theta}{d\rho}$ is negative the subtangent is negative and is measured to the left of the observer.

**6.** Find $\psi$ in the curves $\rho^n = a^n \sin n\theta$ and $\rho^n = b^n \cos n\theta$.

*Ans.* $\psi = n\theta$ and $\frac{\pi}{2} + n\theta$.

**7.** Show that the curves in the preceding example intersect at right angles.

*Hint.* Find $\tau$ for each curve and compare.

**8.** Prove that the spiral of Archimedes $\rho = a\theta$, and the reciprocal spiral $\rho = \frac{a}{\theta}$, intersect at right angles.

**9.** Find the angle between the parabola $\rho = a \sec^2 \frac{\theta}{2}$ and the straight line $\rho \sin \theta = 2a$.

*Ans.* 45°.

**10.** Show that the two cardioids $\rho = a(1 + \cos\theta)$ and $\rho = a(1 - \cos\theta)$ cut each other perpendicularly.

**11.** Find lengths of subtangent, subnormal, tangent, and normal of the spiral of Archimedes $\rho = a\theta$.

*Ans.* subt. $= \frac{\rho^2}{a}$, tan. $= \frac{\rho}{a}\sqrt{a^2 + \rho^2}$,

subn. $= a$, nor. $= \sqrt{a^2 + \rho^2}$.

The student should note the fact that the subnormal is constant.

**12.** Get lengths of subtangent, subnormal, tangent, and normal in the logarithmic spiral $\rho = e^{a\theta}$.

*Ans.* subt. $= \frac{\rho}{\log a}$, tan. $= \rho\sqrt{1 + \frac{1}{\log^2 a}}$,

subn. $= \rho \log a$, nor. $= \rho\sqrt{1 + \log^2 a}$.

When $a = e$ we notice that subt. = subn., and tan. = nor.

**13.** Find the angles between the curves $\rho = a(1 + \cos\theta)$, $\rho = b(1 - \cos\theta)$.

*Ans.* 0 and $\frac{\pi}{2}$.

**14.** Show that the reciprocal spiral $\rho = \frac{a}{\theta}$ has a constant subtangent.

**15.** Prove that the curves $\rho^n = a^n \cos(n\theta - \alpha)$ and $\rho^n = a^n \cos(n\theta - \beta)$ intersect at an angle $\alpha - \beta$.

**16.** Show that the area of the circumscribed square about the cardioid

$$\rho = a(1 - \cos\theta)$$

formed by tangents inclined 45° to the axis is $\frac{27}{16}(2 + \sqrt{3})a^2$.

**82. Solution of equations having multiple roots.** Any root which occurs more than once in an equation is called a *multiple root.* Thus 3, 3, 3, $-2$ are the roots of

$$(A) \qquad x^4 - 7x^3 + 9x^2 + 27x - 54 = 0;$$

hence 3 is a multiple root occurring three times.

Evidently $(A)$ may also be written in the form

$$(x - 3)^3(x + 2) = 0.$$

Let $f(x)$ denote an integral rational function of $x$ having a multiple root $a$, and suppose it occurs $m$ times. Then we may write

$$(B) \qquad f(x) = (x-a)^m \phi(x),$$

where $\phi(x)$ is the product of the factors corresponding to all the roots of $f(x)$ differing from $a$. Differentiating $(B)$,

$$f'(x) = (x-a)^m \phi'(x) + \phi(x) m (x-a)^{m-1}, \text{ or,}$$

$$(C) \qquad f'(x) = (x-a)^{m-1} [(x-a)\phi'(x) + \phi(x) m].$$

Therefore $f'(x)$ contains the factors $(x-a)$ repeated $m-1$ times and no more; that is, the highest common factor (H.C.F.) of $f(x)$ and $f'(x)$ has $m-1$ roots equal to $a$.

In case $f(x)$ has a second multiple root $\beta$ occurring $r$ times, it is evident that the H.C.F. would also contain the factor $(x-\beta)^{r-1}$, and so on for any number of different multiple roots, each occurring once more in $f(x)$ than in the H.C.F.

We may then state a rule for finding the multiple roots of an equation $f(x) = 0$ as follows:

**First step.** *Find $f'(x)$.*

**Second step.** *Find the H.C.F. of $f(x)$ and $f'(x)$.*

**Third step.** *Find the roots of the H.C.F. Each different root of the H.C.F. will occur once more in $f(x)$ than it does in the H.C.F.*

If it turns out that the H.C.F. does not involve $x$, then $f(x)$ has no multiple roots and the above process is of no assistance in the solution of the equation, but it may be of interest to know that the equation has no *equal*, i.e. *multiple*, roots.

Ex. 1. Solve the equation $x^3 - 8x^2 + 13x - 6 = 0$.

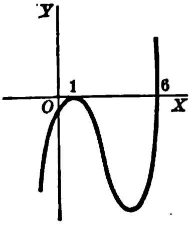
*Solution.* Place $f(x) = x^3 - 8x^2 + 13x - 6$.

*First step.* $f'(x) = 3x^2 - 16x + 13$.

*Second step.* H.C.F. $= x - 1$.

*Third step.* $x - 1 = 0. \quad \therefore x = 1$.

Since 1 occurs once as a root in the H.C.F. it will occur twice in the given equation; that is, $(x-1)^2$ will occur there as a factor. Dividing $x^3 - 8x^2 + 13x - 6$ by $(x-1)^2$ gives the only remaining factor $(x-6)$, yielding the root 6. The roots of our equation are then 1, 1, 6. Drawing the graph of the function, we see that at the double root $x = 1$ the graph touches $OX$ but does not cross it.*



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[*] Since the first derivative vanishes for every multiple root, it follows that the axis of $X$ is tangent to the graph at all points corresponding to multiple roots. If a multiple root occurs an even number of times, the graph will not cross the axis of $X$ at such a point (see figure); if it occurs an odd number of times, the graph will cross.

### EXAMPLES

Solve the first ten equations by the method of this section.

1. $x^3 - 7x^2 + 16x - 12 = 0$. *Ans.* 2, 2, 3.

2. $x^4 - 6x^2 - 8x - 3 = 0$. *Ans.* $-1, -1, -1, 3$.

3. $x^4 - 7x^3 + 9x^2 + 27x - 54 = 0$. *Ans.* $3, 3, 3, -2$.

4. $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$. *Ans.* $3, 3, 3, -4$.

5. $x^4 + 6x^3 + x^2 - 24x + 16 = 0$. *Ans.* $1, 1, -4, -4$.

6. $x^4 - 9x^3 + 23x^2 - 3x - 36 = 0$. *Ans.* $3, 3, -1, 4$.

7. $x^4 - 6x^3 + 10x^2 - 8 = 0$. *Ans.* $2, 2, 1 \pm \sqrt{3}$.

8. $x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0$. *Ans.* $-1, -1, -1, 2, 2$.

9. $x^5 - 15x^3 + 10x^2 + 60x - 72 = 0$. *Ans.* $2, 2, 2, -3, -3$.

10. $x^5 - 3x^4 - 5x^3 + 13x^2 + 24x + 10 = 0$. *Ans.* $-1, -1, -1, 3 \pm \sqrt{-1}$.

Show that the following four equations have no multiple (equal) roots.

11. $x^3 + 9x^2 + 2x - 48 = 0$.

12. $x^4 - 15x^2 - 10x + 24 = 0$.

13. $x^4 - 3x^3 - 6x^2 + 14x + 12 = 0$.

14. $x^n - a^n = 0$.

15. Show that the condition that the equation

$$x^3 + 3qx + r = 0$$

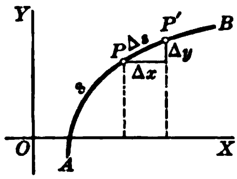
shall have a double root is $4q^3 + r^2 = 0$.

16. Show that the condition that the equation

$$x^3 + 3px^2 + r = 0$$

shall have a double root is $r(4p^3 + r) = 0$.

**83. Applications of the derivative in mechanics. Velocity.** Consider the motion of a point $P$ describing a curve $AB$. Let $s$ be the distance measured along its path from some fixed point as $A$ to any position of $P$, and let $t$ be the corresponding elapsed time. To each value of $t$ corresponds a position of $P$ in the path and therefore a distance (or space) $s$. Hence $s$ will be a function of $t$, and we may write



$$s = f(t).$$

Now let $t$ take on an increment $\Delta t$; then $s$ takes on an increment $\Delta s$, and

$$(A) \qquad \frac{\Delta s}{\Delta t} = \textit{magnitude of the average velocity}^{*}$$

of $P$ during the time interval $\Delta t$. If $P$ moves with uniform motion, the above ratio will have the same value for every interval of time and is the *speed* (= *magnitude of the velocity*) at any instant.

For the general case of any kind of motion, uniform or not, we define the *speed* $v$ (= *magnitude of the velocity*) at any instant as the limit of the ratio $\frac{\Delta s}{\Delta t}$ as $\Delta t$ approaches the limit zero; that is,

$$v = \underset{\Delta t = 0}{\text{limit}} \frac{\Delta s}{\Delta t}, \text{ or,}$$

$$(9) \qquad v = \frac{ds}{dt}.$$

*The speed (= magnitude of the velocity) for any motion is the derivative of the distance (= space) with respect to the time.*

To show that this agrees with the conception we already have of speed, let us find the speed (= magnitude of the velocity) of a falling body at the end of two seconds.

By experiment it has been found that a body falling freely from rest in a vacuum near the earth's surface follows approximately the law

$$(B) \qquad s = 16.1\,t^2,$$

where $s$ = space fallen in feet, $t$ = time in seconds. Apply the *General Rule*, p. 42, to (*B*).

*First step.* $s + \Delta s = 16.1\,(t + \Delta t)^2 = 16.1\,t^2 + 32.2\,t \cdot \Delta t + 16.1\,(\Delta t)^2$.

*Second step.* $\Delta s = 32.2\,t \cdot \Delta t + 16.1\,(\Delta t)^2$.

*Third step.* $\frac{\Delta s}{\Delta t} = 32.2\,t + 16.1\,\Delta t =$ *average speed* (= *magnitude of the average velocity*) *throughout the time interval* $\Delta t$ *reckoned from any fixed instant of time.*†

Placing $t = 2$,

$$(C) \qquad \frac{\Delta s}{\Delta t} = 64.4 + 16.1\,\Delta t = \textit{average speed throughout the}$$

*time interval* $\Delta t$ *after two seconds of falling.*

---

[*] *Velocity* is defined as the time rate of change of place, and is a vector quantity.

[†] $\Delta s$ being the space or distance passed over in the time $\Delta t$.

Our notion of speed tells us at once that (*C*) does not give us the actual speed *at the end of two seconds;* for even if we take $\Delta t$ very small, say $\frac{1}{100}$ or $\frac{1}{1000}$ of a second, (*C*) still gives only the *average speed* during the corresponding small interval of time. But what we do mean by the speed at the end of two seconds is *the limit of the average speed when $\Delta t$ diminishes towards zero;* that is, the speed at the end of two seconds is, from (*C*), 64.4 ft. per second. Thus even the everyday notion of speed which we get from experience involves the idea of a limit, or in our notation

$$v = \underset{\Delta t = 0}{\text{limit}} \left(\frac{\Delta s}{\Delta t}\right) = 64.4 \text{ ft. per second.}$$

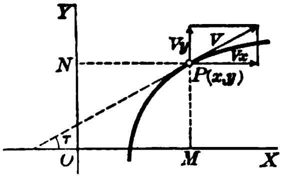
The above example illustrates well the notion of a limiting value. The student should be impressed with the idea that a *limiting value* is a *definite, fixed* value, not something that is only approximated. Observe that it does not make any difference how small $16.1 \, \Delta t$ may be taken; it is only the *limiting value* of

$$64.4 + 16.1 \, \Delta t,$$

when $\Delta t$ diminishes towards zero, that is of importance, and that value is *exactly* 64.4.

**84. Component velocities.** The coördinates $x$ and $y$ of a point $P$ moving in the $XY$ plane are also functions of the time, and the motion may be defined by means of two equations,

$$x = f(t), \qquad y = \phi(t).^*$$



These are the parametric equations of the path (see § 79, p. 92).

The horizontal component $v_x$ of $v$† is the velocity along $OX$ of the projection $M$ of $P$, and is therefore the time rate of change of $x$. Hence, from (9), p. 103, when $s$ is replaced by $x$, we get

$$(10) \qquad v_x = \frac{dx}{dt}.$$

In the same way we get the vertical component, or time rate of change of $y$,

$$(11) \qquad v_y = \frac{dy}{dt}.$$

---

[*] The equation of the path in rectangular coördinates may be found by eliminating $t$ between these equations.

† The direction of $v$ is along the tangent to the path.

Representing the velocity and its components by vectors, we have at once from the figure

$$v^2 = v_x^2 + v_y^2, \text{ or,}$$

(12) $$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

giving the speed (= magnitude of velocity) at any instant.

If $\tau$ be the angle which the direction of the velocity makes with the axis of $X$, we have from the figure, using (9), (10), (11),

(13) $$\sin\tau = \frac{v_y}{v} = \frac{\frac{dy}{dt}}{\frac{ds}{dt}};\ \cos\tau = \frac{v_x}{v} = \frac{\frac{dx}{dt}}{\frac{ds}{dt}};\ \tan\tau = \frac{v_y}{v_x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

**85. Acceleration.** In general $v$ will be a function of $t$ and we may write

$$v = \psi(t).$$

Now let $t$ take on an increment $\Delta t$, then $v$ takes on an increment $\Delta v$, and

$\frac{\Delta v}{\Delta t}$ = *magnitude of the average acceleration** of $P$ during the time interval $\Delta t$.

For any kind of motion we define the magnitude of the acceleration $a$ at any instant as the limit of the ratio $\frac{\Delta v}{\Delta t}$ as $\Delta t$ approaches the limit zero; that is,

$$a = \underset{\Delta t = 0}{\text{limit}} \left(\frac{\Delta v}{\Delta t}\right), \text{ or,}$$

(14) $$a = \frac{dv}{dt}.$$

*The magnitude of the acceleration for any motion is the derivative of the velocity with respect to the time.*

**86. Component accelerations.** Following the same plan used in finding the component velocities, we get for the *component accelerations* parallel to $OX$ and $OY$,

(15) $$a_x = \frac{dv_x}{dt};\ a_y = \frac{dv_y}{dt}. \text{ Also,}$$

(16) $$a = \frac{dv}{dt} = \sqrt{\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2},$$

giving the magnitude of the acceleration.

---

* Acceleration is defined as the time rate of change of velocity, and is a vector quantity.

## EXAMPLES

1. By experiment it has been found that a body falling freely from rest in a vacuum near the earth's surface follows approximately the law

$$s = 16.1\, t^2,$$

where $s$ = space (height) in feet, $t$ = time in seconds. Find magnitudes of the velocity and acceleration

(a) at any instant;

(b) at end of the first second;

(c) at end of the fifth second.

*Solution.* $(A)\qquad s = 16.1\, t^2.$

(a) Differentiating, $\dfrac{ds}{dt} = 32.2\, t$, or, from (9),

$(B)\qquad v = 32.2\, t$ ft. per sec.

Differentiating again, $\dfrac{dv}{dt} = 32.2$, or, from (14),

$(C)\qquad a = 32.2$ ft. per (sec.)$^2$,

which tells us that the acceleration of a falling body is constant; in other words, the velocity increases 32.2 ft. per sec. every second it keeps on falling.

(b) To find $v$ and $a$ at the end of the first second, substitute $t = 1$ in $(B)$ and $(C)$;

$v = 32.2$ ft. per sec.,

$a = 32.2$ ft. per (sec.)$^2$.

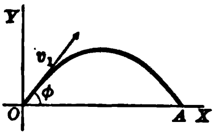
(c) To find $v$ and $a$ at the end of the fifth second, substitute $t = 5$ in $(B)$ and $(C)$;

$v = 161$ ft. per sec.,

$a = 32.2$ ft. per (sec.)$^2$.

2. Neglecting the resistance of the air, the equations of motion for a projectile are

$$x = v_1 \cos\phi \cdot t,$$
$$y = v_1 \sin\phi \cdot t - 16.1\, t^2;$$



where $v_1$ = initial velocity, $\phi$ = angle of projection with horizon, $t$ = time of flight in seconds, $x$ and $y$ being measured in feet. Find the magnitudes of velocity, acceleration, component velocities, component accelerations

(a) at any instant;

(b) at the end of the first second, having given $v_1 = 100$ ft. per sec., $\phi = 30°$;

(c) find direction of motion at the end of the first second.

*Solution.* From (10) and (11),

(a) $\qquad v_x = v_1 \cos\phi;\quad v_y = v_1 \sin\phi - 32.2\, t.$

Also, from (12), $\quad v = \sqrt{v_1^2 - 64.4\, t\, v_1 \sin\phi + 1036.8\, t^2}.$

From (15) and (16), $\quad a_x = 0;\quad a_y = -32.2;\quad a = -32.2.$

(b) Substituting $\quad t = 1,\ v_1 = 100,\ \phi = 30°$ in these results, we get

$v_x = 86.6$ ft. per sec. $\qquad a_x = 0.$

$v_y = 17.8$ ft. per sec. $\qquad a_y = -32.2$ ft. per (sec.)$^2$.

$v = 88.4$ ft. per sec. $\qquad a = -32.2$ ft. per (sec.)$^2$.

(c) $\qquad \tau = \arc\tan \dfrac{v_y}{v_x} = \arc\tan \dfrac{17.8}{86.6} = 11° 36'.9$ = angle of direction of motion with the horizontal.

**3.** If a projectile be given an initial velocity of 200 ft. per sec. in a direction inclined 45° with the horizontal, find

(a) the magnitude of the velocity and direction of motion at the end of the third and sixth seconds;

(b) the component velocities at the same instants.

Conditions are the same as for Ex. 2.

*Ans.* (a) When $t = 3$, $v = 148.3$ ft. per sec., $\tau = 17° 35'$,
when $t = 6$, $v = 150.5$ ft. per sec., $\tau = 159° 53'$;
(b) when $t = 3$, $v_x = 141.4$ ft. per sec., $v_y = 44.8$ ft. per sec.
when $t = 6$, $v_x = 141.4$ ft. per sec., $v_y = 51.8$ ft. per sec.

**4.** The height ($= s$) in feet reached in $t$ seconds by a body projected vertically upwards with a velocity of $v_1$ ft. per sec. is given by the formula

$$s = v_1 t - 16.1\,t^2.$$

Find (a) velocity and acceleration at any instant; and, if $v_1 = 300$ ft. per sec., find velocity and acceleration (b) at end of 2 seconds; (c) at end of 15 seconds. Resistance of air is neglected.

*Ans.* (a) $v = v_1 - 32.2\,t$, $a = -32.2$;
(b) $v = 235.6$ ft. per sec. upwards,
$a = 32.2$ ft. per $(\text{sec.})^2$ downwards;
(c) $v = 183$ ft. per sec. downwards,
$a = 32.2$ ft. per $(\text{sec.})^2$ downwards.

**5.** A cannon ball is fired vertically upwards with a muzzle velocity of 644 ft. per sec. Find (a) its velocity at the end of 10 seconds; (b) for how long it will continue to rise. Conditions same as for Ex. 4.

*Ans.* (a) 322 ft. per sec. upwards;
(b) 20 seconds.

**6.** A train left a station and in $t$ hours was at a distance (space) of

$$s = t^3 + 2\,t^2 + 3\,t$$

miles from the starting point. Find its acceleration* (a) at the end of $t$ hours; (b) at the end of 2 hours.

*Ans.* (a) $a = 3\,t^2 + 4\,t + 3$;
(b) $a = 16$ miles per $(\text{hour})^2$.

**7.** In $t$ hours a train had reached a point at the distance of $\frac{1}{4}t^4 - 4\,t^3 + 16\,t^2$ miles from the starting point. (a) Find its velocity and acceleration. (b) When will the train stop to change the direction of its motion? (c) Describe the motion during the first 10 hours.

*Ans.* (a) $v = t^3 - 12\,t^2 + 32\,t$, $a = 3\,t^2 - 24\,t + 32$;
(b) at end of fourth and eighth hours;
(c) forward first 4 hours, backward the next 4 hours, forward again after 8 hours.

**8.** The space in feet described in $t$ seconds by a point is expressed by the formula

$$s = 48\,t - 16\,t^2.$$

Find the velocity and acceleration at the end of 2½ seconds.

*Ans.* $v = 0$, $a = -32$ ft. per $(\text{sec})^2$.

---

* In this and the following examples the *magnitudes* only of velocity and acceleration are required.

9. Given $s = 2t + 3t^2 + 4t^3$ ft.; find velocity and acceleration (a) at origin; (b) at end of 5 seconds. *Ans.* (a) $v = 2$ ft. per sec., $a = 6$ ft. per (sec.)$^2$;
(b) $v = 332$ ft. per sec., $a = 126$ ft. per (sec.)$^2$.

10. Given $s = \frac{a}{t} + bt^2$, where $a$ and $b$ are constants; find velocity and acceleration at any instant. *Ans.* $v = -\frac{a}{t^2} + 2bt$, $a = \frac{2a}{t^3} + 2b$.

11. At the end of $t$ seconds a body has a velocity of $3t^2 + 2t$ ft. per sec.; find its acceleration (a) in general; (b) at the end of 4 seconds.
*Ans.* (a) $a = 6t + 2$ ft. per (sec.)$^2$; (b) $a = 26$ ft. per (sec.)$^2$.

12. The vertical component of velocity of a point at the end of $t$ seconds is

$$v_y = 3t^2 - 2t + 6 \text{ ft. per sec.}$$

Find the vertical component of acceleration (a) at any instant; (b) at the end of 2 seconds. *Ans.* (a) $a_y = 6t - 2$; (b) 10 ft. per (sec.)$^2$.

13. If a point moves in a fixed path so that

$$s = \sqrt{t},$$

show that the acceleration is negative and proportional to the cube of the velocity.

14. If the distance in feet described by a point in $t$ seconds is given by the formula

$$s = 10 \log \frac{4}{4+t},$$

find velocity and acceleration (a) at the end of 1 second; (b) at the end of 16 seconds.
*Ans.* (a) $v = -\ 2$ ft. per sec., $a = \frac{2}{5}$ ft. per (sec.)$^2$;
(b) $v = -32$ ft. per sec., $a = \frac{1}{40}$ ft. per (sec.)$^2$.

15. If the space described is given by

$$s = ae^t + be^{-t},$$

show that the acceleration is always equal to the space passed over.

16. Given $s = a \cos \frac{\pi t}{2}$; find acceleration. *Ans.* $a = -\frac{\pi^2 s}{4}$.

17. If a point referred to rectangular coördinates moves so that

$$x = a \cos t + b, \text{ and } y = a \sin t + c,$$

show that its velocity has a constant magnitude.

18. If the path of a moving point is the sine curve

$$\begin{cases} x = at, \\ y = b \sin at, \end{cases}$$

show (a) that the $x$ component of the velocity is constant; (b) that the acceleration of the point at any instant is proportional to its distance from the axis of $X$.

19. If a particle moves so that

$$x = t^2,\ y = t^3,$$

(a) show that the path is the semicubical parabola $y^2 = x^3$;
(b) find $v_x$, $v_y$, $v$;
(c) find $a_x$, $a_y$, $a$;
(d) when $t = 2$ sec., find $v$, $a$, position of point (coördinates), and direction of motion.

# CHAPTER VIII

## SUCCESSIVE DIFFERENTIATION

**87. Definition of successive derivatives.** We have seen that the derivative of a function of $x$ is in general also a function of $x$. This new function may also be differentiable, in which case the derivative of the *first derivative* is called the *second derivative* of the original function. Similarly the derivative of the second derivative is called the *third derivative;* and so on to the *n*th *derivative.* Thus, if

$$y = 3x^4,$$

$$\frac{dy}{dx} = 12x^3,$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 36x^2,$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = 72x, \text{ etc.}$$

**88. Notation.** The symbols for the successive derivatives are usually abbreviated as follows:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}.$$

If $y = f(x)$, the successive derivatives are also denoted by

$$f'(x),\ f''(x),\ f'''(x),\ f^{\text{iv}}(x),\ \cdots,\ f^{(n)}(x);$$

$$y',\ y'',\ y''',\ y^{\text{iv}},\ \cdots,\ y^{(n)};$$

or,

$$\frac{d}{dx}f(x),\ \frac{d^2}{dx^2}f(x),\ \frac{d^3}{dx^3}f(x),\ \frac{d^4}{dx^4}f(x),\ \cdots,\ \frac{d^n}{dx^n}f(x).$$

**89. The $n$th derivative.** For certain functions a general expression involving $n$ may be found for the $n$th *derivative*. The usual plan is to find a number of the first successive derivatives, as many as may be necessary to discover their law of formation, and then by induction write down the $n$th derivative.

Ex. 1. Given $y = e^{ax}$; find $\frac{d^ny}{dx^n}$.

*Solution.*

$$\frac{dy}{dx} = ae^{ax},$$

$$\frac{d^2y}{dx^2} = a^2e^{ax},$$

$$\frac{d^3y}{dx^3} = a^3e^{ax},$$

$$\cdot \quad \cdot \quad \cdot$$

$$\therefore \quad \frac{d^ny}{dx^n} = a^ne^{ax}. \quad \textit{Ans.}$$

Ex. 2. Given $y = \log x$; find $\frac{d^ny}{dx^n}$.

*Solution.*

$$\frac{dy}{dx} = \frac{1}{x},$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2},$$

$$\frac{d^3y}{dx^3} = \frac{1 \cdot 2}{x^3},$$

$$\frac{d^4y}{dx^4} = -\frac{1 \cdot 2 \cdot 3}{x^4},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore \quad \frac{d^ny}{dx^n} = (-1)^{n-1}\frac{\underline{|n-1}}{x^n}. \quad \textit{Ans.}$$

Ex. 3. Given $y = \sin x$; find $\frac{d^ny}{dx^n}$.

*Solution.*

$$\frac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2}\right),$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\sin\left(x + \frac{2\pi}{2}\right) = \cos\left(x + \frac{2\pi}{2}\right) = \sin\left(x + \frac{3\pi}{2}\right),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore \quad \frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right). \quad \textit{Ans.}$$

**90. Leibnitz's formula for the $n$th derivative of a product.** This formula expresses the $n$th derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

If $u$ and $v$ are functions of $x$, we have, from V,

$$\frac{d}{dx}(uv)=\frac{du}{dx}v+u\frac{dv}{dx}.$$

Differentiating again with respect to $x$,

$$\begin{aligned}\frac{d^2}{dx^2}(uv)&=\frac{d^2u}{dx^2}v+\frac{du}{dx}\frac{dv}{dx}+\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}\\&=\frac{d^2u}{dx^2}v+2\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{d^3}{dx^3}(uv)&=\frac{d^3u}{dx^3}v+\frac{d^2u}{dx^2}\frac{dv}{dx}+2\frac{d^2u}{dx^2}\frac{dv}{dx}+2\frac{du}{dx}\frac{d^2v}{dx^2}+\frac{du}{dx}\frac{d^2v}{dx^2}+u\frac{d^3v}{dx^3}\\&=\frac{d^3u}{dx^3}v+3\frac{d^2u}{dx^2}\frac{dv}{dx}+3\frac{du}{dx}\frac{d^2v}{dx^2}+u\frac{d^3v}{dx^3}.\end{aligned}$$

However far this process may be continued, it will be seen that the numerical coefficients follow the same law as those of the Binomial Theorem, and the indices of the derivatives correspond to the exponents of the Binomial Theorem.* Reasoning then by mathematical induction from the $m$th to the $(m+1)$th derivative of the product, we can prove *Leibnitz's Formula*

$$\text{(17)}\quad \begin{aligned}\frac{d^n}{dx^n}(uv)=\frac{d^nu}{dx^n}v+n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx}+\frac{n(n-1)}{\underline{|2}}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}+\cdots\\+n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}}+u\frac{d^nv}{dx^{n-1}}.\end{aligned}$$

Ex. 1. Given $y=e^x\log x$; find $\frac{d^3y}{dx^3}$ by Leibnitz's Formula.

*Solution.* Let $u=e^x$, and $v=\log x$;

then

$$\frac{du}{dx}=e^x,\quad \frac{dv}{dx}=\frac{1}{x},$$

$$\frac{d^2u}{dx^2}=e^x,\quad \frac{d^2v}{dx^2}=-\frac{1}{x^2},$$

$$\frac{d^3u}{dx^3}=e^x,\quad \frac{d^3v}{dx^3}=\frac{2}{x^3}.$$

---

* To make this correspondence complete, $u$ and $v$ are considered as $\frac{d^0u}{dx^0}$ and $\frac{d^0v}{dx^0}$.

Substituting in (17), we get

$$\frac{d^3y}{dx^3} = e^x \log x + \frac{3e^x}{x} - \frac{3e^x}{x^2} + \frac{2e^x}{x^3}$$

$$= e^x\left(\log x + \frac{3}{x} - \frac{3}{x^2} + \frac{2}{x^3}\right).$$

**Ex. 2.** Given $y = x^2e^{ax}$; find $\frac{d^ny}{dx^n}$ by Leibnitz's Formula.

*Solution.* Let $u = x^2$, and $v = e^{ax}$;

then

$$\frac{du}{dx} = 2x, \quad \frac{dv}{dx} = ae^{ax},$$

$$\frac{d^2u}{dx^2} = 2, \quad \frac{d^2v}{dx^2} a^2e^{ax},$$

$$\frac{d^3u}{dx^3} = 0, \quad \frac{d^3v}{dx^3} = a^3e^{ax},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^nu}{dx^n} = 0, \quad \frac{d^nv}{dx^n} = a^ne^{ax}.$$

Substituting in (17), we get

$$\frac{d^ny}{dx^n} = x^2a^ne^{ax} + 2a^{n-1}xe^{ax} + n(n-1)a^{n-2}e^{ax}$$

$$= a^{n-2}e^{ax}[x^2a^2 + 2ax + n(n-1)].$$

**91. Successive differentiation of implicit functions.** To illustrate the process we shall find $\frac{d^2y}{dx^2}$ from the equation of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2.$$

Differentiating with respect to $x$ as in § 75, p. 84,

$$2b^2x - 2a^2y\frac{dy}{dx} = 0, \text{ or,}$$

$$(A) \qquad \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Differentiating again, remembering that $y$ is a function of $x$,

$$\frac{d^2y}{dx^2} = \frac{a^2yb^2 - b^2xa^2\frac{dy}{dx}}{a^4y^2}.$$

Substituting for $\frac{dy}{dx}$ its value from $(A)$,

$$\frac{d^2y}{dx^2} = \frac{a^2b^2y - a^2b^2x\left(\frac{b^2x}{a^2y}\right)}{a^4y^2} = -\frac{b^2(b^2x^2 - a^2y^2)}{a^4y^3}.$$

But from the given equation, $b^2x^2 - a^2y^2 = a^2b^2$.

$$\therefore \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

### EXAMPLES

Differentiate the following.

1. $y = 4x^3 - 6x^2 + 4x + 7.$ $\qquad \frac{d^2y}{dx^2} = 12(2x - 1).$

2. $f(x) = \frac{x^3}{1 - x}.$ $\qquad f^{\text{iv}}(x) = \frac{\underline{|4}}{(1 - x)^5}.$

3. $f(y) = y^6.$ $\qquad f^{\text{vi}}(y) = \underline{|6}.$

4. $y = x^3 \log x.$ $\qquad \frac{d^4y}{dx^4} = \frac{6}{x}.$

5. $y = \frac{c}{x^n}.$ $\qquad \frac{d^2y}{dx^2} = \frac{n(n + 1)c}{x^{n+2}}.$

6. $y = (x - 3)e^{2x} + 4xe^x + x.$ $\qquad \frac{d^2y}{dx^2} = 4e^x[(x - 2)e^x + x + 2].$

7. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$ $\qquad \frac{d^2y}{dx^2} = \frac{1}{2a}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) = \frac{y}{a^2}.$

8. $f(x) = ax^2 + bx + c.$ $\qquad f'''(x) = 0.$

9. $f(x) = \log(x + 1).$ $\qquad f^{\text{v}}(x) = -\frac{6}{(x + 1)^4}.$

10. $f(x) = \log(e^x + e^{-x}).$ $\qquad f'''(x) = -\frac{8(e^x - e^{-x})}{(e^x + e^{-x})^3}.$

11. $r = \sin a\theta.$ $\qquad \frac{d^4r}{d\theta^4} = a^4 \sin a\theta = a^4r.$

12. $r = \tan \phi.$ $\qquad \frac{d^3r}{d\phi^3} = 6 \sec^4 \phi - 4 \sec^2 \phi.$

13. $r = \log \sin \phi.$ $\qquad \frac{d^3r}{d\phi^3} = 2 \cot \phi \operatorname{cosec}^2 \phi.$

14. $f(t) = e^{-t} \cos t.$ $\qquad f^{\text{iv}}(t) = -4e^{-t} \cos t = -4f(t).$

15. $f(\theta) = \sqrt{\sec 2\theta}.$ $\qquad f''(\theta) = 3[f(\theta)]^5 - f(\theta).$

16. $p = (q^2 + a^2)\operatorname{arc}\tan\frac{q}{a}$. $\qquad \frac{d^3p}{dq^3} = \frac{4a^3}{(a^2+q^2)^2}$.

17. $y = a^x$. $\qquad \frac{d^ny}{dx^n} = (\log a)^n a^x$.

18. $y = \log(1+x)$. $\qquad \frac{d^ny}{dx^n} = (-1)^{n-1}\frac{|\underline{n-1}}{(1+x)^n}$.

19. $y = \cos ax$. $\qquad \frac{d^ny}{dx^n} = a^n\cos\left(ax + \frac{n\pi}{2}\right)$.

20. $y = x^{n-1}\log x$. $\qquad \frac{d^ny}{dx^n} = \frac{|\underline{n-1}}{x}$.

[$n$ = a positive integer.]

21. $y = \frac{1-x}{1+x}$. $\qquad \frac{d^ny}{dx^n} = 2(-1)^n\frac{|\underline{n}}{(1+x)^{n+1}}$.

*Hint.* Reduce fraction to form $-1+\frac{2}{1+x}$ before differentiating.

22. If $y = e^x\sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

23. If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Use Leibnitz's Formula in the next four examples.

24. $y = x^2a^x$. $\qquad \frac{d^ny}{dx^n} = a^x(\log a)^{n-2}[(x\log a + n)^2 - n]$.

25. $y = e^xx$. $\qquad \frac{d^ny}{dx^n} = e^x(x+n)$.

26. $f(x) = e^x\sin x$. $\qquad f^{(n)}(x) = (\sqrt{2})^n e^x\sin\left(x + \frac{n\pi}{4}\right)$.

27. $f(\theta) = \cos a\theta\cos b\theta$. $\qquad f^{(n)}(\theta) = \frac{(a+b)^n}{2}\cos\left[(a+b)\theta + \frac{n\pi}{2}\right]$

$$+\frac{(a-b)^n}{2}\cos\left[(a-b)\theta + \frac{n\pi}{2}\right].$$

28. Show that the formulas for acceleration, (14), (15), p. 105, may be written

$$a = \frac{d^2s}{dt^2},\quad a_x = \frac{d^2x}{dt^2},\quad a_y = \frac{d^2y}{dt^2}.$$

29. $y^2 = 4ax$. $\qquad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}$.

30. $b^2x^2 + a^2y^2 = a^2b^2$. $\qquad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3};\ \frac{d^3y}{dx^3} = -\frac{3b^6x}{a^4y^5}$.

31. $x^2 + y^2 = r^2$. $\qquad \frac{d^2y}{dx^2} = -\frac{r^2}{y^3}$.

32. $y^2 + y = x^2$. $\qquad \frac{d^3y}{dx^3} = -\frac{24x}{(1+2y)^5}$.

**33.** $ax^2 + 2hxy + by^2 = 1.$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx + by)^3}.$

**34.** $y^2 - 2xy = a^2.$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{a^2}{(y - x)^3};\ \dfrac{d^3y}{dx^3} = -\dfrac{3a^2x}{(y - x)^5}.$

**35.** $\sec\phi \cos\theta = c.$ $\qquad \dfrac{d^2\theta}{d\phi^2} = \dfrac{\tan^2\theta - \tan^2\phi}{\tan^3\theta}.$

**36.** $\theta = \tan(\phi + \theta).$ $\qquad \dfrac{d^3\theta}{d\phi^3} = -\dfrac{2(5 + 8\theta^2 + 3\theta^4)}{\theta^8}.$

**37.** $\log(u + v) = u - v.$ $\qquad \dfrac{d^2v}{du^2} = \dfrac{4(u + v)}{(u + v + 1)^3}.$

**38.** $e^u + u = e^v + v.$ $\qquad \dfrac{d^2v}{du^2} = \dfrac{(e^{u+v} - 1)(e^u - e^v)}{(e^v + 1)^3}.$

**39.** $s = 1 + te^s.$ $\qquad \dfrac{d^2s}{dt^2} = \dfrac{3 - s}{(2 - s)^3}e^{2s}.$

**40.** $e^s + st - e = 0.$ $\qquad \dfrac{d^2s}{dt^2} = s\dfrac{(2 - s)e^s + 2t}{(e^s + t)^3}.$

**41.** $y^3 + x^3 - 3axy = 0.$ $\qquad \dfrac{d^2y}{dx^2} = -\dfrac{2a^3xy}{(y^2 - ax)^3}.$

**42.** $y^2 - 2mxy + x^2 - a = 0.$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{a(m^2 - 1)}{(y - mx)^3}.$

**43.** $y = \sin(x + y).$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{-y}{[1 - \cos(x + y)]^3}.$

**44.** $e^{x+y} = xy.$ $\qquad \dfrac{d^2y}{dx^2} = -\dfrac{y[(x - 1)^2 + (y - 1)^2]}{x^2(y - 1)^3}.$

**45.** $ax^2 + 2hxy + by^2 + 2gx + 2fy + e = 0.$

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}.$$

# CHAPTER IX

## MAXIMA AND MINIMA*

**92. Increasing and decreasing functions.** A function is said to be *increasing* when it increases as the variable increases and decreases as the variable decreases. A function is said to be *decreasing* when it decreases as the variable increases and increases as the variable decreases.

The graph of a function indicates plainly whether it is increasing or decreasing. For instance, consider the function $a^x$ whose graph (Fig. $a$) is the locus of the equation

$$y = a^x. \qquad a > 1$$

As we move along the curve from left to right the curve is *rising*, i.e. as $x$ increases the function $(= y)$ always increases. Therefore $a^x$ is an increasing function for all values of $x$.

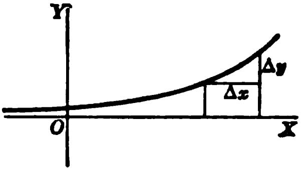


FIG. $a$

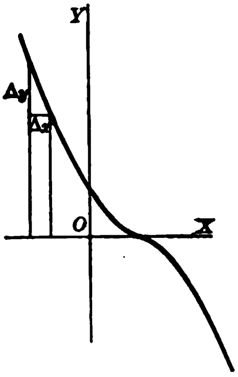


FIG. $b$

On the other hand, consider the function $(a - x)^3$ whose graph (Fig. $b$) is the locus of the equation

$$y = (a - x)^3.$$

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[*] The proofs given in this chapter depend chiefly on geometric intuition. The subject of Maxima and Minima will be treated analytically in § 119, p. 169.

Now as we move along the curve from left to right the curve is *falling*, i.e. as $x$ increases the function $(= y)$ always decreases. Hence $(a-x)^3$ is a decreasing function for all values of $x$.

That a function may be sometimes increasing and sometimes decreasing is shown by the graph (Fig. $c$) of

$$y = 2x^3 - 9x^2 + 12x - 3.$$

As we move along the curve from left to right the curve rises until we reach the point $A$, then it falls from $A$ to $B$, and to the right of $B$ it is always rising. Hence

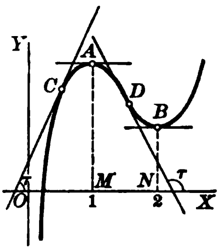


FIG. $c$

(*a*) *from* $x = -\infty$ *to* $x = 1$ *the function is increasing;*

(*b*) *from* $x = 1$ *to* $x = 2$ *the function is decreasing;*

(*c*) *from* $x = 2$ *to* $x = +\infty$ *the function is increasing.*

The student should study the curve carefully in order to note the behavior of the function when $x = 1$ and $x = 2$. Evidently $A$ and $B$ are turning points. At $A$ the function ceases to increase and commences to decrease; at $B$, the reverse is true. At $A$ and $B$ the tangent (or curve) is evidently parallel to the axis of $X$, and therefore the slope is zero.

**93. Tests for determining when a function is increasing and when decreasing.** It is evident from Fig. $c$ that at a point, as $C$, where a function

$$y = f(x)$$

is *increasing*, the tangent in general makes an acute angle with the axis of $X$; hence

$$slope = tan\ \tau = \frac{dy}{dx} = f'(x) = a\ positive\ number.$$

Similarly at a point, as $D$, where a function is *decreasing*, the tangent in general makes an obtuse angle with the axis of $X$; therefore

$$slope = tan\ \tau = \frac{dy}{dx} = f'(x) = a\ negative\ number.^*$$

---

[*] Conversely, for any given value of $x$,

*if* $f'(x) = +$, *then* $f(x)$ *is increasing;*
*if* $f'(x) = -$, *then* $f(x)$ *is decreasing.*

When $f'(x) = 0$, we cannot decide without further investigation whether $f(x)$ is increasing or decreasing.

In order then that the function shall change from an increasing to a decreasing function, or vice versa, it is a necessary and sufficient condition that **the first derivative shall change sign.** But this can only happen for a continuous derivative by passing through the value zero. Thus in Fig. *c*, p. 117, as we pass along the curve the derivative (= slope) changes sign at $A$ and $B$ where it has the value zero. In general then we have at **turning points**

$$\frac{dy}{dx} = f'(x) = 0. \tag{18}$$

The derivative is continuous in nearly all our important applications, but it is interesting to note the case when the derivative (= slope) changes sign by passing through ∞.* This would evidently happen at the points $B$, $E$, $G$ in Fig. *d*, p. 119, where the tangents (and curve) are perpendicular to the axis of $X$. At such exceptional turning points

$$\frac{dy}{dx} = f'(x) = \infty\,;$$

or, what amounts to the same thing,

$$\frac{1}{f'(x)} = 0.$$

**94. Maximum and minimum values of a function.**† A *maximum* value of a function is one that is *greater* than any values immediately preceding or following.

A *minimum* value of a function is one that is *less* than any values immediately preceding or following.

For example, in Fig. *c*, p. 117, it is clear that the function has a maximum value $MA\,(= y = 2)$ when $x = 1$, and a minimum value $NB\,(= y = 1)$ when $x = 2$.

The student should observe that a maximum value is not necessarily the greatest possible value of a function nor a minimum value the least. For, in Fig. *c* it is seen that the function (= $y$) has values to the right of $B$ that are greater than the maximum $MA$, and values to the left of $A$ that are less than the minimum $NB$.

---

* By this is meant that its reciprocal passes through the value zero.

† The student should not forget that in general the definitions and proofs given in this book apply only at points where the function is continuous.

A function may have several maximum and minimum values. Suppose that the following figure represents the graph of a function $f(x)$.

At $B$, $D$, $G$, $I$, $K$ the function is a maximum, and at $C$, $E$, $H$, $J$ a minimum. That some particular minimum value of a function may be greater than some particular maximum value is shown in the figure, the minimum values at $C$ and $H$ being greater than the maximum value at $K$.

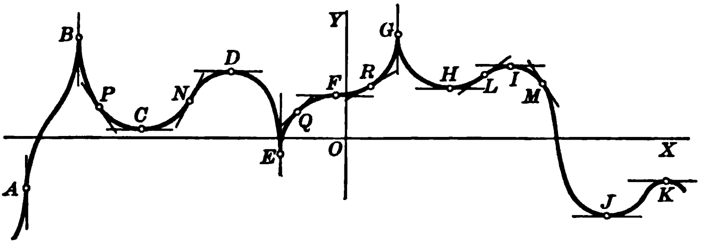


Fig. *d*

At the ordinary turning points $C$, $D$, $H$, $I$, $J$, $K$ the tangent (or curve) is parallel to $OX$; therefore

$$\textit{slope} = \frac{dy}{dx} = f'(x) = 0.$$

At the exceptional turning points $B$, $E$, $G$ the tangent (or curve) is perpendicular to $OX$, giving

$$\textit{slope} = \frac{dy}{dx} = f'(x) = \infty.$$

One of these two conditions is then necessary in order that the function shall have a maximum or a minimum value. But such a condition is not sufficient; for, at $F$ the slope is zero and at $A$ it is infinite, and yet the function has neither a maximum nor a minimum value at either point. It is necessary for us to know, in addition, how the function behaves in the neighborhood of each point. Thus at the points of *maximum value*, $B$, $D$, $G$, $I$, $K$, the function *changes from an increasing to a decreasing function*, and at the points of *minimum value*, $C$, $E$, $H$, $J$, the function *changes* *from a decreasing to an increasing function.* It therefore follows from § 93 that at *maximum points*

$$slope = \frac{dy}{dx} = f'(x) \text{ must change from } + \text{ to } -,$$

and at *minimum points*

$$slope = \frac{dy}{dx} = f'(x) \text{ must change from } - \text{ to } +$$

when we move along the curve from left to right.

At such points as $A$ and $F$ where the slope is zero or infinite, but which are *neither maximum nor minimum points*,

$$slope = \frac{dy}{dx} = f'(x) \text{ does not change sign.}$$

We may then state the conditions in general for maximum and minimum values of $f(x)$ for certain values of the variable as follows:

**(19) $f(x)$ is a maximum if $f'(x) = 0$, and $f'(x)$ changes from $+$ to $-$.**

**(20) $f(x)$ is a minimum if $f'(x) = 0$, and $f'(x)$ changes from $-$ to $+$.**

The values of the variable at the turning points of a function are called *critical values;* thus $x = 1$ and $x = 2$ are the critical values of the variable for the function whose graph is shown in Fig. *c*, p. 117. The critical values at turning points where the tangent is parallel to $OX$ are evidently found by placing the first derivative equal to zero and solving for real values of $x$, just as under § 77, p. 86.*

To determine the sign of the first derivative at points near a particular turning point, substitute in it, first a value of the variable just a *little less* than the corresponding critical value, and then one a *little greater.*† If the first gives $+$ (as at $L$, Fig. *d*, p. 119) and the second $-$ (as at $M$), then the function ($= y$) has a maximum value in that interval (as at $I$).

---

* Similarly if we wish to examine a function at exceptional turning points where the tangent is perpendicular to $OX$, we set the reciprocal of the first derivative equal to zero and solve to find critical values.

† In this connection the term "little less," or "trifle less," means any value between the next smaller root (critical value) and the one under consideration; and the term "little greater," or "trifle greater," means any value between the root under consideration and the next larger one.

If the first gives $-$ (as at $P$) and the second $+$ (as at $N$), then the function ($= y$) has a minimum value in that interval (as at $C$).

If the sign is the same in both cases (as at $Q$ and $R$), then the function ($= y$) has neither a maximum nor a minimum value in that interval (as at $F$).*

We shall now summarize our results into a compact *working rule.*

**95. First method for examining a function for maximum and minimum values. Working rule.**

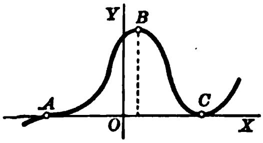
**First step.** *Find the first derivative of the function.*

**Second step.** *Set the first derivative equal to zero*† *and solve the resulting equation for real roots in order to find the critical values of the variable.*

**Third step.** *Write the derivative in factor form; if it is algebraic, write it in linear factor form.*

**Fourth step.** *Considering one critical value at a time, test the first derivative, first for a value a trifle less and then for a value a trifle greater than the critical value. If the sign of the derivative is first* $+$ *and then* $-$, *the function has a maximum value for that particular critical value of the variable; but if the reverse is true, then it has a minimum value. If the sign does not change, the function has neither.*

Ex. 1. Examine the function $(x-1)^2(x+1)^3$ for maximum and minimum values.



*Solution.* $f(x) = (x-1)^2(x+1)^3.$

*First step.* $f'(x) = 2(x-1)(x+1) + 3(x-1)^2(x+1)^2$

$= (x-1)(x+1)^2(5x-1).$

*Second step.* $(x-1)(x+1)^2(5x-1) = 0,$

$x = 1,\ -1,\ \frac{1}{5}$, which are critical values.

*Third step.* $f'(x) = 5(x-1)(x+1)^2(x-\frac{1}{5}).$

*Fourth step.* Examine first for critical value $x = 1$ ($C$ in figure).

When $x < 1$, $f'(x) = 5(-)(+)^2(+) = -.$

When $x > 1$, $f'(x) = 5(+)(+)^2(+) = +.$

Therefore, when $x = 1$ the function has a minimum value $f(1) = 0$ ($=$ ordinate of $C$).

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* A similar discussion will evidently hold for the exceptional turning points $B$, $E$, and $A$ respectively.

† When the first derivative becomes infinite for a certain value of the independent variable, then the function should be examined for such a critical value of the variable, for it may give maximum or minimum values, as at $B$, $E$, or $A$ (Fig. $d$, p. 119). See footnote on preceding page.

Examine now for critical value $x = \frac{1}{3}$ ($B$ in figure).

$$\text{When } x < \tfrac{1}{3},\ f'(x) = 5(-)(+)^2(-) = +.$$
$$\text{When } x > \tfrac{1}{3},\ f'(x) = 5(-)(+)^2(+) = -.$$

Therefore, when $x = \frac{1}{3}$ the function has a maximum value $f(\frac{1}{3}) = 1.11 +$ (= ordinate of $B$).

Examine lastly for critical value $x = -1$ ($A$ in figure).

$$\text{When } x < -1,\ f'(x) = 5(-)(-)^2(-) = +.$$
$$\text{When } x > -1,\ f'(x) = 5(-)(+)^2(-) = +.$$

Therefore, when $x = -1$ the function has neither a maximum nor a minimum value.

Ex. 2. Examine $\sin x(1 + \cos x)$ for maximum and minimum values.

*Solution.* $f(x) = \sin x(1 + \cos x)$.

*First step.* $f'(x) = -\sin^2 x + (1 + \cos x)\cos x = 2\cos^2 x + \cos x - 1.$

*Second step.* $2\cos^2 x + \cos x - 1 = 0.$

Solving the quadratic, $\cos x = \frac{1}{2}$ or $-1$;

hence the critical values are $x = \pm\frac{\pi}{3}$ or $\pi$.

*Third step.* $f'(x) = 2(\cos x - \frac{1}{2})(\cos x + 1).$

*Fourth step.* Examine first for critical value $x = \frac{\pi}{3}$.

$$\text{When } x < \frac{\pi}{3},\ f'(x) = 2(+)(+) = +.$$
$$\text{When } x > \frac{\pi}{3},\ f'(x) = 2(-)(+) = -.$$

Therefore, when $x = \frac{\pi}{3}$ the function has a maximum value $f\left(\frac{\pi}{3}\right) = \frac{3}{4}\sqrt{3}$.

Examine now for critical value $x = -\frac{\pi}{3}$.

$$\text{When } x < -\frac{\pi}{3},\ f'(x) = 2(-)(+) = -.$$
$$\text{When } x > -\frac{\pi}{2},\ f'(x) = 2(+)(+) = +.$$

Therefore, when $x = -\frac{\pi}{3}$ the function has a minimum value $f\left(-\frac{\pi}{3}\right) = -\frac{3}{4}\sqrt{3}$.

Examine now for critical value $x = \pi$.

$$\text{When } x < \pi,\ f'(x) = 2(-)(+) = -.$$
$$\text{When } x > \pi,\ f'(x) = 2(-)(+) = -.$$

Therefore, when $x = \pi$ the function has neither a maximum nor a minimum value.

Since the cosine is a periodic function, the critical values are really

$$x = 2n\pi \pm \frac{\pi}{3} \text{ and } n\pi,$$

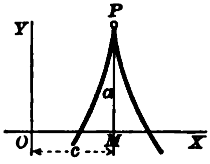
where $n$ is any integer. Therefore the function has an infinite number of maxima all equal to $\frac{3}{4}\sqrt{3}$, and an infinite number of minima all equal to $-\frac{3}{4}\sqrt{3}$.

Ex. 3. Examine the function $a - b(x - c)^{\frac{2}{3}}$ for maxima and minima.

*Solution.*

$$f(x) = a - b(x - c)^{\frac{2}{3}}.$$

$$f'(x) = -\frac{2b}{3(x - c)^{\frac{1}{3}}}.$$



Since $x = c$ is a critical value for which $f'(x) = \infty$, but for which $f(x)$ is not infinite, let us test the function for maximum and minimum values when $x = c$.

When $x < c$, $f'(x) = +$.

When $x > c$, $f'(x) = -$.

Hence, when $x = c = OM$ the function has a maximum value $f(c) = a = MP$.

**96. Second method for examining a function for maximum and minimum values.** From (19), p. 120, it is clear that in the vicinity of a maximum value of $f(x)$, in passing along the graph from left to right,

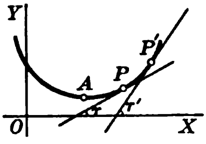
$$f'(x) \text{ changes from } + \text{ to } 0 \text{ to } -.^{*}$$

Hence $f'(x)$ is a decreasing function, and by § 93 we know that its derivative, i.e. the second derivative of the function itself $[=f''(x)]$, is negative or zero.

Similarly we have, from (20), p. 120, that in the vicinity of a minimum value of $f(x)$

$$f'(x) \text{ changes from } - \text{ to } 0 \text{ to } +.$$

Hence $f'(x)$ is an increasing function, and by § 93 it follows that $f''(x)$ is positive or zero.



The student should observe that $f''(x)$ is positive not only at minimum points (as at $A$) but also at points such as $P$. For, as a point passes through $P$ in moving from left to right,

$$slope = \tan\tau = \frac{dy}{dx} = f'(x) \text{ is an increasing function.}$$

At such a point the curve is said to be *concave upwards.*

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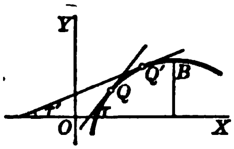
* $f(x)$ is assumed to be continuous, and $f''(x)$ to exist.



Similarly $f''(x)$ is negative not only at maximum points (as at $B$) but also at points such as $Q$. For, as a point passes through $Q$,

$$\textit{slope} = \tan\tau = \frac{dy}{dx} = f'(x) \textit{ is a decreasing function.}$$

At such a point the curve is said to be *concave downwards.*[*]

We may then state the sufficient conditions for maximum and minimum values of $f(x)$ for certain values of the variable as follows:

**(21) $f(x)$ is a maximum if $f'(x) = 0$ and $f''(x) =$ a negative number.**

**(22) $f(x)$ is a minimum if $f'(x) = 0$ and $f''(x) =$ a positive number.**

Following is the corresponding **working rule.**

**First step.** *Find the first derivative of the function.*

**Second step.** *Set the first derivative equal to zero and solve the resulting equation for real roots in order to find the critical values of the variable.*

**Third step.** *Find the second derivative.*

**Fourth step.** *Substitute each critical value for the variable in the second derivative. If the result is negative, then the function is a maximum for that critical value; if the result is positive, the function is a minimum.*[†]

Ex. 1. Examine $x^3 - 3x^2 - 9x + 5$ for maxima and minima.

*Solution.* $f(x) = x^3 - 3x^2 - 9x + 5.$

*First step.* $f'(x) = 3x^2 - 6x - 9.$

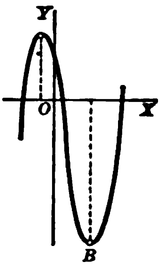
*Second step.* $3x^2 - 6x - 9 = 0;$

hence the critical values are $x = -1$ and $3$.

*Third step.* $f''(x) = 6x - 6.$

*Fourth step.* $f''(-1) = -12.$ $\therefore f(-1) = 10 =$ (ordinate of $A$) = maximum value.

$f''(3) = +12.$ $\therefore f(3) = 22 =$ (ordinate of $B$) = minimum value.



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[*] At a point where the curve is *concave upwards* we sometimes say that the curve has a *positive bending*, and where it is *concave downwards* a *negative bending*.

[†] When $f''(x) = 0$, or does not exist, the above process fails, although there may even then be a maximum or a minimum; in that case the first method given in the last section still holds, being fundamental. Usually this second method does apply, and when the process of finding the second derivative is not too long or tedious, it is generally the shortest method.

**Ex. 2.** Examine $\sin^2 x \cos x$ for maximum and minimum values.

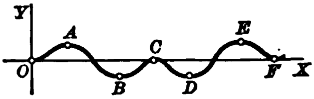
*Solution.* $f(x) = \sin^2 x \cos x$.

*First step.* $f'(x) = 2 \sin x \cos^2 x - \sin^3 x$.

*Second step.* $2 \sin x \cos^2 x - \sin^3 x = 0$;

hence the critical values are $x = n\pi$

and $x = n\pi \pm \text{arc} \tan \sqrt{2} = n\pi \pm a$.



*Third step.* $f''(x) = \cos x (2 \cos^2 x - 7 \sin^2 x)$.

*Fourth step.* $f''(0) = +$. $\therefore f(0) = 0 =$ minimum value at $O$.

$f''(\pi) = -$. $\therefore f(\pi) = 0 =$ maximum value at $C$.

$f''(a) = -$. $\therefore f(a) =$ maximum value at $A$.

$f''(\pi - a) = +$. $\therefore f(\pi - a) =$ minimum value at $B$, etc.

The work of finding maximum and minimum values may frequently be simplified by the aid of the following principles which follow at once from our discussion of the subject.

(1) *The maximum and minimum values of a continuous function must occur alternately.*

(2) *When c is a positive constant, $cf(x)$ is a maximum or a minimum for such values of x, and such only, as make $f(x)$ a maximum or a minimum.*

Hence, in determining the critical values of $x$ and testing for maxima and minima, any constant factor may be omitted.

*When c is negative, $cf(x)$ is a maximum when $f(x)$ is a minimum, and conversely.*

(3) *If c is a constant,*

$$f(x) \text{ and } c + f(x)$$

*have maximum and minimum values for the same values of x.*

Hence a constant term may be omitted when finding critical values of $x$ and testing.

## EXAMPLES

Examine the following functions for maximum and minimum values.

1. $3x^3 - 9x^2 - 27x + 30$. *Ans.* $x = -1$, gives max. $= 45$;
$x = 3$, gives min. $= -51$.

2. $2x^3 - 21x^2 + 36x - 20$. *Ans.* $x = 1$, gives max. $= -3$;
$x = 6$, gives min. $= -128$.

3. $\frac{x^3}{3} - 2x^2 + 3x + 1$. *Ans.* $x = 1$, gives max. $= \frac{7}{3}$;
$x = 3$, gives min. $= 1$.

4. $2x^3 - 15x^2 + 36x + 10$. *Ans.* $x = 2$, gives max. $= 38$;
$x = 3$, gives min. $= 37$.

5. $x^3 - 9x^2 + 15x - 3$. *Ans.* $x = 1$, gives max. $= 4$;
$x = 5$, gives min. $= -28$.

6. $x^3 - 3x^2 + 6x + 10$. *Ans.* No max. or min.

7. $x^5 - 5x^4 + 5x^3 + 1$. *Ans.* $x = 1$, gives max. $= 2$;
$x = 3$, gives min. $= -26$;
$x = 0$, gives neither.

8. $3x^5 - 125x^3 + 2160x$. *Ans.* $x = -4$ and $3$, give max.;
$x = -3$ and $4$, give min.

9. $(x-3)^2(x-2)$. *Ans.* $x = \frac{7}{3}$, gives max. $= \frac{4}{27}$;
$x = 3$, gives min. $= 0$.

10. $(x-1)^3(x-2)^2$. *Ans.* $x = \frac{8}{5}$, gives max. $= .03456$;
$x = 2$, gives min. $= 0$;
$x = 1$, gives neither.

11. $(x-4)^5(x+2)^4$. *Ans.* $x = -2$, gives max.;
$x = \frac{2}{3}$, gives min.;
$x = 4$, gives neither.

12. $(x-2)^5(2x+1)^4$. *Ans.* $x = -\frac{1}{2}$, gives max.;
$x = \frac{11}{18}$, gives min.;
$x = 2$, gives neither.

13. $(x+1)^{\frac{2}{3}}(x-5)^2$. *Ans.* $x = \frac{1}{2}$, gives max.;
$x = -1$ and $5$, give min.

14. $(2x-a)^{\frac{1}{3}}(x-a)^{\frac{2}{3}}$. *Ans.* $x = \frac{2a}{3}$, gives max.;
$x = a$, gives min.;
$x = \frac{a}{2}$, gives neither.

15. $x(x-1)^2(x+1)^3$. *Ans.* $x = \frac{1}{2}$, gives max.;
$x = 1$ and $-\frac{1}{3}$, give min.

16. $x(a+x)^2(a-x)^3$. *Ans.* $x = -a$ and $\frac{a}{3}$, give max.;
$x = -\frac{a}{2}$, gives min.

17. $b + c(x-a)^{\frac{2}{3}}$. *Ans.* $x = a$, gives min. $= b$.

18. $a - b(x-c)^{\frac{1}{3}}$. *Ans.* No max. or min.

19. $\frac{x^2 - 7x + 6}{x - 10}$. *Ans.* $x = 4$, gives max.;
$x = 16$, gives min.

20. $\frac{(a-x)^3}{a-2x}$. *Ans.* $x = \frac{a}{4}$, gives min.

21. $\frac{1 - x + x^2}{1 + x - x^2}$. *Ans.* $x = \frac{1}{2}$, gives min.

22. $\frac{x^2 - 3x + 2}{x^2 + 3x + 2}$. *Ans.* $x = \sqrt{2}$, gives min. $= 12\sqrt{2} - 17$;
$x = -\sqrt{2}$, gives max. $= -12\sqrt{2} - 17$.

23. $\frac{(x-a)(b-x)}{x^2}$. *Ans.* $x = \frac{2ab}{a+b}$, gives max. $= \frac{(a-b)^2}{4ab}$.

24. $\frac{a^2}{x} + \frac{b^2}{a-x}$. *Ans.* $x = \frac{a^2}{a-b}$, gives min.;
$x = \frac{a^2}{a+b}$, gives max.

25. $\frac{x}{\log x}$. *Ans.* $x = e$, gives min.

26. $\frac{e^x}{x} + e^{-2x}$. *Ans.* Min. when $x$ lies between $\frac{12}{11}$ and $\frac{13}{12}$.

27. $ae^{kx} + be^{-kx}$. *Ans.* Min. $= 2\sqrt{ab}$.

28. $x^x$. *Ans.* $x = \frac{1}{e}$, gives min.

29. $x^{\frac{1}{x}}$. *Ans.* $x = e$, gives max.

30. $\cos x + \sin x$. *Ans.* $x = \frac{\pi}{4}$, gives max. $= \sqrt{2}$;
$x = \frac{5\pi}{4}$, gives min. $= -\sqrt{2}$.

31. $\sin 2x - x$. *Ans.* $x = \frac{\pi}{6}$, gives max.;
$x = -\frac{\pi}{6}$, gives min.

32. $x + \tan x$. *Ans.* No max. or min.

33. $\sin^3 x \cos x$. *Ans.* $x = \frac{\pi}{3}$, gives max. $= \frac{3}{16}\sqrt{3}$;
$x = -\frac{\pi}{3}$, gives min. $= -\frac{3}{16}\sqrt{3}$.

34. $x \cos x$. *Ans.* $x = \cot x$, gives max.

35. $\sin x + \cos 2x$. *Ans.* $x = \text{arc} \sin \frac{1}{4}$, gives max.;
$x = \frac{\pi}{2}$, gives min.

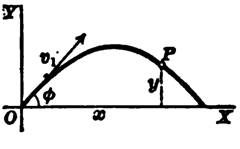
36. $2 \tan x - \tan^2 x$. *Ans.* $x = \frac{\pi}{4}$, gives max.

37. $\frac{\sin x}{1 + \tan x}$. *Ans.* $x = \frac{\pi}{4}$, gives max.

38. $\frac{x}{1 + x \tan x}$. *Ans.* $x = \cos x$, gives max.

39. The range $OA$ of a projectile in a vacuum is given by the formula

$$R = \frac{v_1^2 \sin 2\phi}{g};$$



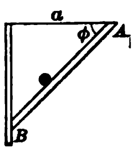
where $v_1$ = initial velocity, $g$ = acceleration due to gravity, $\phi$ = angle of projection with the horizontal. Find the angle of projection which gives the greatest range for a given initial velocity. *Ans.* $\phi = 45°$.

40. The total time of flight of the projectile in the last problem is given by the formula

$$T = \frac{2 v_1 \sin \phi}{g}.$$

At what angle should it be projected in order to make the time of flight a maximum? *Ans.* $\phi = 90°$.

41. The time it takes a ball to roll down an inclined plane $AB$ is given by the formula



$$T = 2\sqrt{\frac{a}{g \sin 2\phi}}.$$

Neglecting friction, etc., what must be the value of $\phi$ to make the quickest descent? *Ans.* $\phi = 45°$.

42. When the resistance of air is taken into account, the inclination of a pendulum to the vertical may be given by the formula

$$\theta = ae^{-kt}\cos(nt + \epsilon).$$

Show that the greatest elongations occur at equal intervals $\frac{\pi}{n}$ of time.

43. It is required to measure a certain unknown magnitude $x$ with precision. Suppose that $n$ equally careful observations of the magnitude are made, giving the results

$$a_1, a_2, a_3, \cdots, a_n.$$

The errors of these observations are evidently

$$x - a_1, x - a_2, x - a_3, \cdots, x - a_n,$$

some of which are positive and some negative.

It has been agreed that the most probable value of $x$ is such that it renders the sum of the squares of the errors, namely

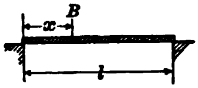
$$(x - a_1)^2 + (x - a_2)^2 + (x - a_3)^2 + \cdots + (x - a_n)^2,$$

a minimum. Show that this gives the arithmetical mean of the observations as the most probable value of $x$.

44. The bending moment at $B$ of a beam of length $l$, uniformly loaded, is given by the formula

$$M = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2,$$

where $w$ = load per unit length. Show that the maximum bending moment is at the center of the beam.



45. If the total waste per mile in an electric conductor is

$$W = c^2r + \frac{t^2}{r},$$

where $c$ = current in amperes, $r$ = resistance in ohms per mile, and $t$ = a constant depending on the interest on the investment and the depreciation of the plant, what is the relation between $c$, $r$, and $t$ when the waste is a minimum? *Ans.* $cr = t$.

46. A submarine telegraph cable consists of a core of copper wires with a covering made of nonconducting material. If $x$ denote the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signaling varies as

$$x^2 \log \frac{1}{x}.$$

Show that the greatest speed is attained when $x = \frac{1}{\sqrt{e}}$.

**47.** Assuming that the power given out by a voltaic cell is given by the formula

$$P = \frac{E^2R}{(r + R)^2},$$

where $E$ = constant electro-motive force, $r$ = constant internal resistance, $R$ = external resistance, prove that $P$ is a maximum when $r = R$.

**48.** When a battery of $mn$ cells is joined up so that $m$ rows of $n$ cells, connected in series, are joined in parallel, the current is given by the formula

$$C = \frac{mnE}{mR + nr},$$

where $E$ = electro-motive force of each cell, $r$ = internal, and $R$ = external resistance of each cell. Show that the current is a maximum when $Rm = rn$, that is, the total internal resistance equals the total external resistance.

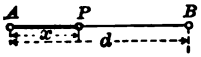
**49.** The force exerted by a circular electric current of radius $a$ on a small magnet whose axis coincides with the axis of the circle varies as

$$\frac{x}{(a^2 + x^2)^{\frac{5}{2}}},$$

where $x$ = distance of magnet from plane of circle. Prove that the force is a maximum when $x = \frac{a}{2}$.

**50.** We have two sources of heat at $A$ and $B$ with intensities $a$ and $b$ respectively. The total intensity of heat at a distance of $x$ from $A$ is given by the formula

$$I = \frac{a}{x^2} + \frac{b}{(d - x)^2}.$$



Show that the temperature at $P$ will be the lowest when

$$\frac{d - x}{x} = \frac{\sqrt[3]{b}}{\sqrt[3]{a}};$$

that is, the distances $BP$ and $AP$ have the same ratio as the cube roots of the corresponding heat intensities. The distance of $P$ from $A$ is

$$x = \frac{a^{\frac{1}{3}}d}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}.$$

**97. General directions for solving problems involving maxima and minima.** In our work thus far the function has been given whose maximum and minimum values were required. This is by no means always the case; in fact, we are generally obliged to construct the function ourselves from the conditions given in the problem, and then test it as we have been doing for maximum and minimum values.

No rule applicable in all cases can be given for constructing the function, but in a large number of problems we may be guided by the following

**General Directions.**

(*a*) *Express the function whose maximum or minimum is involved in the problem.*

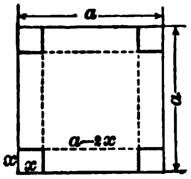
(*b*) *If the resulting expression contains more than one variable, the conditions of the problem will furnish enough relations between the variables so that all may be expressed in terms of a single one.*

(*c*) *To the resulting function of a single variable apply one of our two rules for finding maximum and minimum values.*

(*d*) *In practical problems it is usually easy to tell which critical value will give a maximum and which a minimum value, so it is not always necessary to apply the fourth step of our rules.*

### PROBLEMS

1. It is desired to make an open-top box of greatest possible volume from a square piece of tin whose side is $a$ by cutting equal squares out of the corners and then folding up the tin to form the sides. What should be the length of a side of the squares cut out?



*Solution.* Let $x$ = side of small square = depth of box; then $a - 2x$ = side of square forming bottom of box, and volume is

$$V = (a - 2x)^2 x;$$

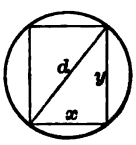
which is the function to be made a maximum by varying $x$. Applying rule,

*First step.* $$\frac{dV}{dx} = (a - 2x)^2 - 4x(a - 2x) = a^2 - 8ax + 12x^2.$$

*Second step.* Solving $a^2 - 8ax + 12x^2 = 0$ gives critical values $x = \frac{a}{2}$ and $\frac{a}{6}$.

It is evident from the figure that $x = \frac{a}{2}$ must give a minimum, for then all the tin would be cut away, leaving no material out of which to make a box. By the usual test, $x = \frac{a}{6}$ is found to give a maximum volume $\frac{2a^3}{27}$. Hence the side of the square to be cut out is one sixth of the side of the given square.

2. Assuming that the strength of a beam with rectangular cross section varies directly as the breadth and as the square of the depth, what are the dimensions of the strongest beam that can be sawed out of a round log whose diameter is $d$?



*Solution.* If $x$ = breadth and $y$ = depth, then the beam will have maximum strength when the function $xy^2$ is a maximum. From the figure, $y^2 = d^2 - x^2$; hence we should test the function

$$f(x) = x(d^2 - x^2).$$

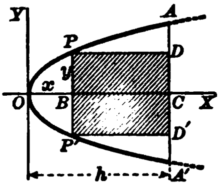
*First step.* $f'(x) = -2x^2 + d^2 - x^2 = d^2 - 8x^2.$

*Second step.* $d^2 - 3x^2 = 0.$ $\therefore x = \dfrac{d}{\sqrt{3}}$ = critical value which gives a maximum.
Therefore, if the beam is cut so that

$$\text{depth} = \sqrt{\tfrac{2}{3}} \text{ of the diameter of log,}$$

and $$\text{breadth} = \sqrt{\tfrac{1}{3}} \text{ of the diameter of log,}$$

the beam will have maximum strength.



3. What is the width of the rectangle of maximum area that can be inscribed in a given segment $OAA'$ of a parabola?

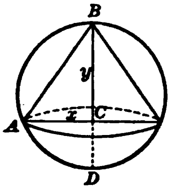
*Hint.* If $OC = h$, $BC = h - x$ and $PP' = 2y$; therefore the area of rectangle $PDD'P'$ is

$$2(h - x)y.$$

But since $P$ lies on the parabola $y^2 = 2px$, the function to be tested is

$$2(h - x)\sqrt{2px}.$$

*Ans.* Width $= \frac{2}{3}h$.



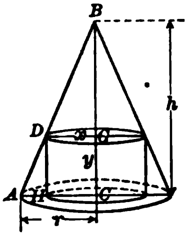
4. Find the altitude of the cone of maximum volume that can be inscribed in a sphere of radius $r$.

*Hint.* Volume of cone $= \frac{1}{3}\pi x^2 y$. But $x^2 = BC \times CD = y(2r - y)$; therefore the function to be tested is

$$f(y) = \frac{\pi}{3} y^2(2r - y).$$

*Ans.* Altitude of cone $= \frac{4}{3}r$.

5. Find the altitude of the cylinder of maximum volume that can be inscribed in a given right cone.



*Hint.* Let $AC = r$ and $BC = h$. Volume of cylinder $= \pi x^2 y$.
But from similar triangles $ABC$ and $DBG$

$$r : x :: h : h - y. \quad \therefore x = \frac{r(h - y)}{h}.$$

Hence the function to be tested is

$$f(y) = y(h - y)^2.$$

*Ans.* Altitude $= \frac{1}{3}h$.

6. Divide $a$ into two parts such that their product is a maximum.

*Ans.* Each part $= \dfrac{a}{2}$.

7. Divide 10 into two parts such that the sum of their squares is a minimum.

*Ans.* Each part = 5.

8. Divide 10 into two such parts that the sum of the square of one and the cube of the other may be a maximum. *Ans.* 4 and 6.

9. Find the number that exceeds its square by the greatest possible quantity.

*Ans.* $\frac{1}{2}$.

10. What number added to its reciprocal gives the least possible sum? *Ans.* 1.

11. Assuming that the stiffness of a beam of rectangular cross section varies directly as the breadth and the cube of the depth, what must be the breadth of the stiffest beam that can be cut from a log 16 inches in diameter?

*Ans.* Breadth = 8 inches.

12. A torpedo boat is anchored 9 miles from the nearest point of a beach, and it is desired to send a messenger in the shortest possible time to a military camp situated 15 miles from that point along the shore. If he can walk 5 miles an hour but row only 4 miles an hour, required the place he must land.

*Ans.* 3 miles from the camp.

13. For a certain specified sum a man takes the contract to build a rectangular water tank lined with lead. It has a square base and open top, and holds 108 cubic yards. What shall be its dimensions in order to require the least possible quantity of lead?

*Ans.* Altitude = 3 yds., side = 6 yds., that is,
length of side = twice the altitude.

14. A gasholder is a cylindrical vessel closed at the top and open at the bottom, where it sinks into the water. What should be its proportions for a given volume to require the least material (this would also give least weight)?

*Ans.* Diameter = double the height.

15. What should be the dimensions and weight of a gasholder of 8,000,000 cubic feet capacity built in the most economical manner out of sheet iron $\frac{1}{16}$ of an inch thick and weighing $2\frac{1}{2}$ lbs. per sq. ft.?

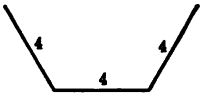
*Ans.* Height = 137 ft., diam. = 273 ft., weight = 200 tons.

16. What are the most economical proportions for a cylindrical steam boiler?

*Ans.* Diameter = length.

17. A paper-box manufacturer has in stock a quantity of strawboard 30 inches by 14 inches. Out of this material he wishes to make open-top boxes by cutting equal squares out of each corner and then folding up to form the sides. Find the side of the square that should be cut out in order to give the boxes maximum volume.

*Ans.* 3 inches.



18. A roofer wishes to make an open gutter of maximum capacity whose bottom and sides are each 4 inches wide and whose sides have the same slope. What should be the width across the top?

*Ans.* 8 inches.

19. Assuming that the energy expended in driving a steamboat through the water varies as the cube of her velocity, find her most economical rate per hour when steaming against a current running $c$ miles per hour.

*Hint.* Let $v$ = most economical speed;
then $av^3$ = energy expended each hour, $a$ being a constant depending upon the particular conditions,
and $v-c$ = actual distance advanced per hour.

Hence $\frac{av^3}{v-c}$ is the energy expended per mile of distance advanced, and it is therefore the function whose minimum is wanted.

*Ans.* $v = \frac{3}{2}c$.

20. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base. Show that when the canvas is laid out flat it will be a circle with a sector of about 66° cut out. A bell tent 10 ft. high should then have a base of diameter 14 ft. and would require 272 sq. ft. of canvas.

**21.** Find the right triangle of maximum area that can be constructed on a line of length $h$ as hypotenuse. *Ans.* $\frac{h}{\sqrt{2}}$ = length of both legs.

**22.** Show that a square is the rectangle of maximum area that can be inscribed in a given circle. Also show that the square has the maximum perimeter.

**23.** What is the isosceles triangle of maximum area that can be inscribed in a given circle? *Ans.* An equilateral triangle.

**24.** Find the altitude of the maximum rectangle that can be inscribed in a right triangle with base $b$ and altitude $h$. *Ans.* Altitude $= \frac{h}{2}$.

**25.** Find the dimensions of the rectangle of maximum area that can be inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. *Ans.* $a\sqrt{2}$ and $b\sqrt{2}$; area $= 2ab$.

**26.** Find the altitude of the right cylinder of maximum volume that can be inscribed in a sphere of radius $r$. *Ans.* Altitude of cylinder $= \frac{2r}{\sqrt{3}}$.

**27.** Find the altitude of the right cylinder of maximum convex (curved) surface that can be inscribed in a given sphere. *Ans.* Altitude of cylinder $= r\sqrt{2}$.

**28.** Find the altitude of the right cylinder inscribed in a given sphere when its entire surface is a maximum. *Ans.* Altitude $= \left(2 - \frac{2}{\sqrt{5}}\right)^{\frac{1}{2}} r$.

**29.** Find the altitude of the right cone inscribed in a sphere when its entire surface is a maximum. *Ans.* Altitude $= (23 - \sqrt{17})\frac{r}{16}$.

**30.** Find the altitude of the right cone of minimum volume circumscribed about a given sphere. *Ans.* Altitude $= 4r$, and volume $= 2 \times$ vol. of sphere.

**31.** A right cone of maximum volume is inscribed in a given right cone, the vertex of the inside cone being at the center of the base of the given cone. Show that the altitude of the inside cone is one third the altitude of the given cone.

**32.** Through a point $(a, b)$ referred to rectangular axes a straight line is to be drawn, forming with the axes a triangle of least area. Show that its intercepts on the axes are $2a$ and $2b$.

**33.** Through the point $(a, b)$ a line is drawn such that the part intercepted between the axes is a minimum. Prove that its length is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

**34.** Given a point on the axis of the parabola $y^2 = 2px$ at a distance $a$ from the vertex; find the abscissa of the point of the curve nearest to it. *Ans.* $x = a - p$.

**35.** What is the length of the shortest line that can be drawn tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and meeting the coördinate axes? *Ans.* $a + b$.

**36.** Find the altitude of the least isosceles triangle that can be circumscribed about an ellipse, the base being parallel to the major axis.

*Ans.* Altitude = 3 times semi-minor axis.

**37.** A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter; required the height and breadth of the window when the quantity of light admitted is a maximum.

*Ans.* Radius of circle = height of rectangle.

**38.** A tapestry 7 feet in height is hung on a wall so that its lower edge is 9 feet above an observer's eye. At what distance from the wall should he stand in order to obtain the most favorable view?

*Hint.* The vertical angle subtended by the tapestry in the eye of the observer must be at a maximum. *Ans.* 12 feet.

**39.** The regulations of the British Parcels Post require that the sum of the length and girth of a parcel shall not exceed 6 feet. Show that

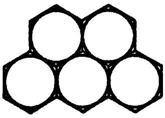
(a) The greatest sphere allowed is about 17⅜ inches in diameter and has a volume of about 1½ cubic feet.

(b) The greatest cube has an edge 14⅖ inches in length and a volume of nearly 1¾ cubic feet.

(c) The greatest rectangular box is 1 ft. by 1 ft. by 2 ft., containing 2 cubic feet.

(d) The greatest parcel of any shape is a cylinder 2 ft. long and 4 ft. circumference, and contains over 2½ cubic feet.

**40.** What are the most economical proportions of a tin can which shall have a given capacity, making allowance for waste?



*Hint.* There is no waste in cutting out tin for the side of the can, but for top and bottom a hexagon of tin circumscribing the circular pieces required is used up.

*Ans.* Height $= \frac{2\sqrt[3]{3}}{\pi} \times$ diameter of base.

NOTE 1. Ex. 16 shows that if no allowance is made for waste, then height = diameter.

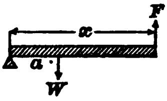
NOTE 2. We know that the shape of a bee cell is hexagonal, giving a certain capacity for honey with the greatest possible economy of wax.

**41.** An open cylindrical trough is constructed by bending a given sheet of tin of breadth $2a$. Find the radius of the cylinder of which the trough forms a part when the capacity of the trough is a maximum.

*Ans.* Rad. $= a$, i.e. bent in the form of a semicircle.

**42.** A weight $W$ is to be raised by means of a lever with the force $F$ at one end and the point of support at the other. If the weight is suspended from a point at a distance $a$ from the point of support, and the weight of the beam is $w$ pounds per linear foot, what should be the length of the lever in order that the force required to lift it shall be a minimum?

*Ans.* $x = \sqrt{\frac{2aW}{w}}$ feet.



**43.** A rectangular stockade is to be built which must have a certain area. If a stone wall already constructed is available for one of the sides, find the dimensions which would make the cost of construction the least.

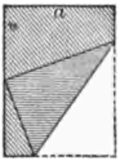
*Ans.* Side parallel to wall = twice the length of each end.

44. An electric arc light is to be placed directly over the center of a circular plot of grass 100 feet in diameter. Assuming that the intensity of light varies directly as the sine of the angle under which it strikes an illuminated surface and inversely as the square of its distance from the surface, how high should the light be hung in order that the best possible light shall fall on a walk along the circumference of the plot?

*Ans.* $\frac{50}{\sqrt{2}}$ feet.

45. The lower corner of a leaf, whose width is $a$, is folded over so as just to reach the inner edge of page. (a) Find the width of the part folded over when the length of the crease is a minimum. (b) Find width when the area folded over is a minimum.

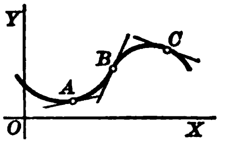
*Ans.* (a) $\frac{3}{4}a$; (b) $\frac{2}{3}a$.



# CHAPTER X

## POINTS OF INFLECTION

**98. Definition.** *Points of inflection* separate arcs concave upwards from arcs concave downwards.* Thus, if a curve $y=f(x)$ changes (as at $B$) from concave upwards (as at $A$) to concave downwards (as at $C$), or the reverse, then such a point as $B$ is called a point of inflection.



From the discussion of § 96 it follows at once that at $A$, $f''(x)=+$, and at $C$, $f''(x)=-$. In order to change sign it must pass through the value zero;† hence we have

**(28)** **At points of inflection, $f''(x)=0$.**

Solving the equation resulting from (28) gives the abscissas of the points of inflection. To determine the direction of curving in the vicinity of a point of inflection, test $f''(x)$ for values of $x$, first a trifle less and then a trifle greater than the abscissa at that point.

If $f''(x)$ changes sign, we have a point of inflection, and the signs obtained determine if the curve is concave upwards or concave downwards in the neighborhood of each point.

The student should observe that near a point where the curve is concave upwards (as at $A$) the curve lies above the tangent, and at a point where the curve is concave downwards (as at $C$) the curve lies below the tangent. At a point of inflection (as at $B$) the tangent evidently crosses the curve.

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* Points of inflection may also be defined as points where

(a) $\dfrac{d^2y}{dx^2}=0$ and $\dfrac{d^2y}{dx^2}$ changes sign, or

(b) $\dfrac{d^2x}{dy^2}=0$ and $\dfrac{d^2x}{dy^2}$ changes sign.

† It is assumed that $f'(x)$ and $f''(x)$ are continuous. The solution of Ex. 2, p. 137, shows how to discuss a case where $f'(x)$ and $f''(x)$ are both infinite. Evidently salient points (see p. 265) are excluded, since at such points $f'(x)$ is discontinuous.

Following is a **rule for finding points of inflection** of the curve whose equation is $y = f(x)$, including also directions for examining the curve in the neighborhood of such a point.

**First step.** *Find* $f''(x)$.

**Second step.** *Set* $f''(x) = 0$, *and solve the equation for real roots.*

**Third step.** *Write* $f''(x)$ *in factor form.*

**Fourth step.** *Test* $f''(x)$ *for values of* $x$, *first a trifle less and then a trifle greater than each root found in the second step. If* $f''(x)$ *changes sign, we have a point of inflection.*

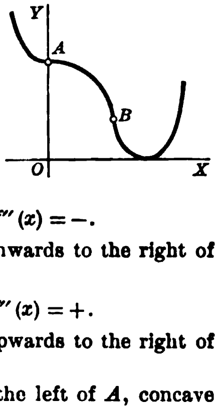
*When* $f''(x) = +$, *the curve is concave upwards* $\overset{+}{\smile}$.*

*When* $f''(x) = -$, *the curve is concave downwards* $\underset{-}{\frown}$.

EXAMPLES

Examine the following curves for points of inflection and direction of bending.

1. $y = 3x^4 - 4x^3 + 1$.



*Solution.* $f(x) = 3x^4 - 4x^3 + 1$.

*First step.* $f''(x) = 36x^2 - 24x$.

*Second step.* $36x^2 - 24x = 0$.

$\therefore x = \frac{2}{3}$ and $x = 0$, critical values.

*Third step.* $f''(x) = 36x(x - \frac{2}{3})$.

*Fourth step.* When $x < 0, f''(x) = +$; and when $x > 0, f''(x) = -$.

$\therefore$ curve is concave upwards to the left and concave downwards to the right of $x = 0$ ($A$ in figure).

When $x < \frac{2}{3}, f''(x) = -$; and when $x > \frac{2}{3}, f''(x) = +$.

$\therefore$ curve is concave downwards to the left and concave upwards to the right of $x = \frac{2}{3}$ ($B$ in figure).

The curve is evidently concave upwards everywhere to the left of $A$, concave downwards between $A$ $(0, 1)$ and $B$ $(\frac{2}{3}, \frac{11}{27})$, and concave upwards everywhere to the right of $B$.

2. $(y - 2)^3 = (x - 4)$.

*Solution.* $y = 2 + (x - 4)^{\frac{1}{3}}$.

*First step.*

$$\frac{dy}{dx} = \frac{1}{3}(x - 4)^{-\frac{2}{3}},$$

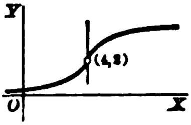
$$\frac{d^2y}{dx^2} = -\frac{2}{3}(x - 4)^{-\frac{5}{3}}.$$

---

* This may be easily remembered if we say that a vessel shaped like the curve where it is concave upwards holds (+) water, and where it is concave downwards spills (−) water.

*Second step.* When $x = 4$, both first and second derivatives are infinite.

*Fourth step.* When $x < 4$, $\frac{d^2y}{dx^2} = +$; but when $x > 4$, $\frac{d^2y}{dx^2} = -$.



We may therefore conclude that the tangent at (4, 2) is perpendicular to the axis of $X$, that to the left of (4, 2) the curve is concave upwards, and to the right of (4, 2) it is concave downwards. Therefore (4, 2) must be considered a point of inflection.

3. $y = x^2$. *Ans.* Concave upwards everywhere.

4. $y = 5 - 2x - x^2$. *Ans.* Concave downwards everywhere.

5. $y = x^3$. *Ans.* Concave downwards to the left and concave upwards to the right of (0, 0).

6. $y = x^3 - 3x^2 - 9x + 9$. *Ans.* Concave downwards to the left and concave upwards to the right of $(1, -2)$.

7. $y = a + (x - b)^3$. *Ans.* Concave downwards to the left and concave upwards to the right of $(a, b)$.

8. $a^2y = \frac{x^3}{3} - ax^2 + 2a^3$. *Ans.* Concave downwards to the left and concave upwards to the right of $\left(a, \frac{4a}{3}\right)$.

9. $x^3 - 3bx^2 + a^2y = 0$. *Ans.* Point of inflection is $\left(b, \frac{2b^3}{a^2}\right)$.

10. $y = x^4$. *Ans.* Concave upwards everywhere.

11. $y = x^4 - 12x^3 + 48x^2 - 50$.
*Ans.* Concave upwards to the left of $x = 2$, concave downwards between $x = 2$ and $x = 4$, concave upwards to the right of $x = 4$.

12. $y = \frac{8a^3}{x^2 + 4a^2}$. *Ans.* Concave downwards between $\left(\pm\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$, concave upwards outside of these points.

13. $y = x + 36x^2 - 2x^3 - x^4$. *Ans.* Points of inflection at $x = 2$ and $-3$.

14. $y = \frac{x^3}{x^2 + 3a^2}$. *Ans.* Concave upwards to the left of $\left(-3a, -\frac{9a}{4}\right)$, concave downwards between $\left(-3a, -\frac{9a}{4}\right)$ and (0, 0), concave upwards between (0, 0) and $\left(3a, \frac{9a}{4}\right)$, concave downwards to the right of $\left(3a, \frac{9a}{4}\right)$.

15. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$. *Ans.* Points of inflection are $x = \pm\frac{a}{\sqrt{2}}$.

**16.** $a^4y^2 = a^2x^4 - x^6$. *Ans.* Points of inflection are $x = \pm\frac{a}{6}\sqrt{27 - 3\sqrt{33}}$.

**17.** $y = \frac{x}{a^2 + x^2}$. *Ans.* Points of inflection are $x = 0, \pm a\sqrt{3}$.

**18.** $y = \sin x$. *Ans.* Points of inflection are $x = n\pi$, $n$ being any integer.

**19.** $y = \tan x$. *Ans.* Points of inflection are $x = n\pi$, $n$ being any integer.

**20.** $y = xe^{-x}$. *Ans.* $x = 2$ gives a point of inflection.

**21.** Show that no conic section can have a point of inflection.

**22.** Show that the graphs of $e^x$ and $\log x$ have no points of inflection.

**23.** Show that the curve $y(x^2 + a^2) = x$ has three points of inflection lying on the straight line $x - 4ay = 0$.

**24.** Show that the abscissas of the points of inflection of the curve $y^2 = f(x)$ satisfy the equation

$$[f'(x)]^2 = 2f(x) \cdot f''(x).$$

# CHAPTER XI

## DIFFERENTIALS

**99. Introduction.** Thus far we have represented the derivative of $y = f(x)$ by the notation

$$\frac{dy}{dx} = f'(x).$$

We have taken special pains to impress on the student that the symbol

$$\frac{dy}{dx}$$

was to be considered not as an ordinary fraction with $dy$ as numerator and $dx$ as denominator, but as a single symbol denoting the limit of the quotient

$$\frac{\Delta y}{\Delta x}$$

as $\Delta x$ approaches the limit zero.

Problems do occur, however, where it is very convenient to be able to give a meaning to $dx$ and $dy$ separately, and it is especially useful in applications of the Integral Calculus. How this may be done is explained in what follows.

**100. Definitions.** If $f'(x)$ is the derivative of $f(x)$ for a particular value of $x$, and $\Delta x$ is an arbitrarily chosen increment of $x$, then the *differential of $f(x)$*, denoted by the symbol $df(x)$, is defined by the equation

$$(A) \qquad df(x) = f'(x)\,\Delta x.$$

If now $f(x) = x$, then $f'(x) = 1$, and $(A)$ reduces to

$$dx = \Delta x,$$

showing that when $x$ is the independent variable the ***differential of $x$*** $(= dx)$ is identical with $\Delta x$. Hence, if $y = f(x)$, $(A)$ may in general be written in the form

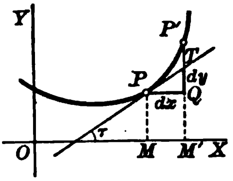
$$(B) \qquad dy = f'(x)\,dx.^*$$

---

[*] On account of the position which the derivative $f'(x)$ here occupies, it is sometimes called the *differential coefficient*.

The student should observe the important fact that, since $dx$ may be given any arbitrary value whatever, *$dx$ is independent of $x$*. Hence *$dy$ is a function of two independent variables $x$ and $dx$.*

***The differential of a function equals its derivative multiplied by the differential of the independent variable.***



Let us illustrate what this means geometrically.

Let $f'(x)$ be the derivative of $y=f(x)$ at $P$.

Take $dx = PQ$, then

$$dy = f'(x)\,dx = \tan\tau \cdot PQ = \frac{QT}{PQ}\cdot PQ = QT.$$

Therefore $dy$, or $df(x)$, is the increment $(= QT)$ of the ordinate of the tangent corresponding to $dx$.[*]

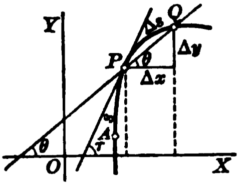
This gives the following interpretation of the derivative as a fraction.

*If an arbitrarily chosen increment of the independent variable $x$ for a point $P\,(x, y)$ on the curve $y=f(x)$ be denoted by $dx$, then in the derivative*

$$\frac{dy}{dx} = f'(x) = \tan\tau$$

*$dy$ denotes the corresponding increment of the tangent to the curve at the point $P$.*

**101. $dx$ and $dy$ considered as infinitesimals.** In the Differential Calculus we are usually concerned with the derivative, i.e. with the ratio of the differentials $dy$ and $dx$. In some applications it is also useful to consider $dx$ as an infinitesimal (see § 30, p. 21). Then by $(B)$, p. 140, and (2), p. 27, $dy$ is also an infinitesimal. Hence in such cases $dx$ and $dy$ are corresponding variables each of which approaches the limit zero.[†]



**102. Derivative of the arc in rectangular coördinates.** Let $s$ be the length[‡] of the arc $AP$ measured from a fixed point $A$ on the curve. Denoting the corresponding increments by $\Delta x$, $\Delta y$, $\Delta s$, we have from the figure

$$(\text{chord } PQ)^2 = (\Delta x)^2 + (\Delta y)^2.$$

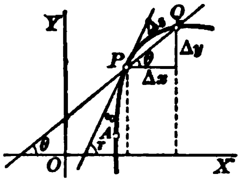
---

[*] The student should note especially that the differential $(=dy)$ and the increment $(=\Delta y)$ of the function corresponding to the same value $dx\,(=\Delta x)$ are not in general equal. For, in the figure, $dy = QT$ but $\Delta y = QP'$.

[†] In the Integral Calculus $dx$ and $dy$ are always regarded as infinitesimals.

[‡] Defined in § 224.

Both multiplying and dividing the first member by (arc $PQ$)$^2$ [$=(\Delta s)^2$], we get



$$(A)\qquad \left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 (\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.$$

Dividing both members by $\Delta x$,

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

From this we get, when $\Delta x$ approaches the limit zero,

$$(B)\qquad \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2,$$

assuming that $\underset{PQ=0}{\text{limit}}\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right) = 1.$ Hence

$$(24)\qquad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Similarly, if we divide $(A)$ by $\Delta y$ and pass to the limit, we get

$$(25)\qquad \frac{ds}{dy} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}.$$

Also, from the above figure,

$$\cos\theta = \frac{\Delta x}{\text{chord } PQ} = \frac{\Delta x}{\Delta s}\cdot\frac{\text{arc } PQ}{\text{chord } PQ}, \text{ and}$$

$$\sin\theta = \frac{\Delta y}{\text{chord } PQ} = \frac{\Delta y}{\Delta s}\cdot\frac{\text{arc } PQ}{\text{chord } PQ}.$$

[Multiplying both numerator and denominator in each case by arc $PQ$ ($=\Delta s$).]

As $\Delta s$ approaches the limit zero, $\theta$ approaches the limit $\tau$, and the ratio of the arc $PQ$ to the chord $PQ$ approaches unity. Therefore

$$(26)\qquad \cos\tau = \frac{dx}{ds},\ \sin\tau = \frac{dy}{ds}.$$

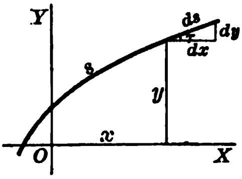
Using the notation of differentials, (24) and (25) may be written

$$(27)\qquad ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx.$$

$$(28)\qquad ds = \left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{\frac{1}{2}} dy.$$

An easy way to remember the relations (24) to (28) between the differentials $dx$, $dy$, $ds$ is to note that they are correctly represented by a right triangle whose hypotenuse is $ds$, sides $dx$ and $dy$, and angle at base $\tau$. Then

$$ds = \sqrt{(dx)^2 + (dy)^2},$$



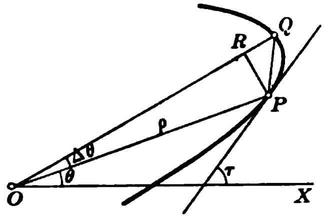
and dividing by $dx$ or $dy$ gives (24) or (25) respectively. Also, from the figure,

$$\cos\tau = \frac{dx}{ds},\ \sin\tau = \frac{dy}{dx};$$

the same relations given by (26).

**103. Derivative of the arc in polar coördinates.** In what follows we shall employ the same figure, notation, and reductions used on pp. 97, 98. From the right triangle $PRQ$

$$\begin{aligned}(\text{chord } PQ)^2 &= (PR)^2 + (RQ)^2 \\ &= (\rho\sin\Delta\theta)^2 + (\rho + \Delta\rho - \rho\cos\Delta\theta)^2 \\ &= \rho^2\sin^2\Delta\theta + \left(2\rho\sin^2\frac{\Delta\theta}{2} + \Delta\rho\right)^2.\end{aligned}$$



Multiplying and dividing the first member by $(\text{arc } PQ)^2$ $[= (\Delta s)^2]$, and then dividing throughout by $(\Delta\theta)^2$, we get

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2\left(\frac{\Delta s}{\Delta\theta}\right)^2 = \rho\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(\rho\sin\frac{\Delta\theta}{2}\cdot\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} + \frac{\Delta\rho}{\Delta\theta}\right)^2.$$

Passing to the limit as $\Delta\theta$ diminishes towards zero (see § 80, p. 98), we get

$$\left(\frac{ds}{d\theta}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2.$$

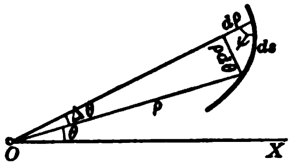
(29) $$\therefore\ \frac{ds}{d\theta} = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}.$$

In the notation of differentials this becomes

(30) $$ds = \left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta.$$

These relations between $\rho$ and the differentials $ds$, $d\rho$, and $d\theta$ are correctly represented by a right triangle whose hypotenuse is $ds$ and sides $d\rho$ and $\rho d\theta$. Then

$$ds = \sqrt{(\rho d\theta)^2 + (d\rho)^2},$$



and dividing by $d\theta$ gives (29).

Denoting by $\psi$ the angle between $d\rho$ and $ds$, we get at once

$$\tan\psi = \rho \frac{d\theta}{d\rho},$$

which is the same as ($A$), p. 98.

**104. Formulas for finding the differentials of functions.** Since the differential of a function is its derivative multiplied by the differential of the independent variable, it follows at once that the formulas for finding differentials are the same as those for finding derivatives given in § 46, pp. 46–48, if we multiply each one by $dx$.

This gives us

I $\qquad d(c) = 0.$

II $\qquad d(x) = dx.$

III $\qquad d(u + v - w) = du + dv - dw.$

IV $\qquad d(cv) = cdv.$

V $\qquad d(uv) = u\,dv + v\,du.$

VI $\qquad d(v_1 v_2 \cdots v_n) = (v_2 v_3 \cdots v_n)\,dv_1 + (v_1 v_3 \cdots v_n)\,dv_2 + \cdots + (v_1 v_2 \cdots v_{n-1})\,dv_n.$

VII $\qquad d(v^n) = nv^{n-1}dv.$

VII *a* $\qquad d(x^n) = nx^{n-1}dx.$

VIII $\qquad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$

VIII *a* $\qquad d\left(\frac{u}{c}\right) = \frac{du}{c}.$

VIII *b* $\qquad d\left(\frac{c}{v}\right) = -\frac{cdv}{v^2}.$

**IX** $$d(\log_a v) = \log_a e \frac{dv}{v}.$$

**X** $$d(a^v) = a^v \log a \, dv.$$

**X *a*** $$d(e^v) = e^v \, dv.$$

**XI** $$d(u^v) = vu^{v-1} du + \log u \cdot u^v \cdot dv.$$

**XII** $$d(\sin v) = \cos v \, dv.$$

**XIII** $$d(\cos v) = -\sin v \, dv.$$

**XIV** $$d(\tan v) = \sec^2 v \, dv, \text{ etc.}$$

**XIX** $$d(\text{arc} \sin v) = \frac{dv}{\sqrt{1 - v^2}}, \text{ etc.}$$

The term *differentiation* also includes the operation of finding differentials.

In finding differentials, the easiest way is to find the derivative as usual, and then multiply the result by $dx$.

**Ex. 1.** Find the differential of

$$y = \frac{x + 3}{x^2 + 3}.$$

*Solution.* $$dy = d\left(\frac{x + 3}{x^2 + 3}\right) = \frac{(x^2 + 3) d(x + 3) - (x + 3) d(x^2 + 3)}{(x^2 + 3)^2}$$

$$= \frac{(x^2 + 3) dx - (x + 3) 2x dx}{(x^2 + 3)^2} = \frac{(3 - 6x - x^2) dx}{(x^2 + 3)^2}. \quad \textit{Ans.}$$

**Ex. 2.** Find $dy$ from

$$b^2x^2 - a^2y^2 = a^2b^2.$$

*Solution.* $$2b^2x dx - 2a^2y dy = 0.$$

$$\therefore dy = \frac{b^2x}{a^2y} dx. \quad \textit{Ans.}$$

**Ex. 3.** Find $d\rho$ from

$$\rho^2 = a^2 \cos 2\theta.$$

*Solution.* $$2\rho d\rho = -a^2 \sin 2\theta \cdot 2 d\theta.$$

$$\therefore d\rho = -\frac{a^2 \sin 2\theta}{\rho} d\theta.$$

**Ex. 4.** Find $d[\text{arc} \sin (3t - 4t^3)]$.

*Solution.* $$d[\text{arc} \sin (3t - 4t^3)] = \frac{d(3t - 4t^3)}{\sqrt{1 - (3t - 4t^3)^2}} = \frac{3 dt}{\sqrt{1 - t^2}}. \quad \textit{Ans.}$$

**105. Successive differentials.** As the differential of a function is in general also a function of the independent variable, we may deal with its differential. Consider the function

$$y = f(x).$$

$d(dy)$ is called the *second differential of* $y$ (or *of the function*) and is denoted by the symbol

$$d^2y.$$

Similarly the *third differential of* $y$, $d[d(dy)]$ is written

$$d^3y,$$

and so on, to the $n$th *differential of* $y$,

$$d^ny.$$

Since $dx$, the differential of the independent variable, is independent of $x$ (see footnote, p. 140), it must be treated as a constant when differentiating with respect to $x$. Bearing this in mind we get very simple relations between *successive differentials* and *successive derivatives*. For

$$dy = f'(x)\,dx,$$

and

$$d^2y = f''(x)\,(dx)^2,$$

since $dx$ is regarded as a constant.

Also,

$$d^3y = f'''(x)\,(dx)^3,$$

and in general

$$d^ny = f^{(n)}(x)\,(dx)^n.$$

Dividing both sides of each expression by the power of $dx$ occurring on the right, we get our ordinary derivative notation

$$\frac{d^2y}{dx^2} = f''(x),\quad \frac{d^3y}{dx^3} = f'''(x),\ \cdots,\ \frac{d^ny}{dx^n} = f^{(n)}(x).$$

**Ex. 1.** Find the third differential of

$$y = x^5 - 2x^3 + 3x - 5.$$

*Solution.*

$$dy = (5x^4 - 6x^2 + 3)\,dx,$$

$$d^2y = (20x^3 - 12x)\,(dx)^2,$$

$$d^3y = (60x^2 - 12)\,(dx)^3.\quad \textit{Ans.}$$

NOTE. This is evidently the derivative of the function multiplied by the cube of the differential of the independent variable. Dividing through by $(dx)^3$, we get the third derivative

$$\frac{d^3y}{dx^3} = 60x^2 - 12.$$

### EXAMPLES

**Differentiate the following, using differentials.**

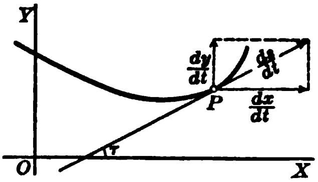
1. $y = ax^3 - bx^2 + cx + d.$ $\qquad dy = (3\,ax^2 - 2\,bx + c)\,dx.$
2. $y = 2\,x^{\frac{5}{2}} - 3\,x^{\frac{2}{3}} + 6\,x^{-1} + 5.$ $\qquad dy = (5\,x^{\frac{3}{2}} - 2\,x^{-\frac{1}{3}} - 6\,x^{-2})\,dx.$
3. $y = (a^2 - x^2)^5.$ $\qquad dy = -\,10\,x\,(a^2 - x^2)^4\,dx.$
4. $y = \sqrt{1 + x^2}.$ $\qquad dy = \dfrac{x}{\sqrt{1 + x^2}}\,dx.$
5. $y = \dfrac{x^{2n}}{(1 + x^2)^n}.$ $\qquad dy = \dfrac{2\,nx^{2n-1}}{(1 + x^2)^{n+1}}\,dx.$
6. $y = \log\sqrt{1 - x^3}.$ $\qquad dy = \dfrac{3\,x^2dx}{2\,(x^3 - 1)}.$
7. $y = (e^x + e^{-x})^2.$ $\qquad dy = 2\,(e^{2x} - e^{-2x})\,dx.$
8. $y = e^x \log x.$ $\qquad dy = e^x\left(\log x + \dfrac{1}{x}\right)dx.$
9. $s = t - \dfrac{e^t - e^{-t}}{e^t + e^{-t}}.$ $\qquad ds = \left(\dfrac{e^t - e^{-t}}{e^t + e^{-t}}\right)^2 dt.$
10. $\rho = \tan\phi + \sec\phi.$ $\qquad d\rho = \dfrac{1 + \sin\phi}{\cos^2\phi}\,d\phi.$
11. $r = \frac{1}{3}\tan^3\theta + \tan\theta.$ $\qquad dr = \sec^4\theta d\theta.$
12. $f(x) = (\log x)^3.$ $\qquad f'(x)\,dx = \dfrac{3\,(\log x)^2dx}{x}.$
13. $\phi(t) = \dfrac{t^3}{(1 - t^2)^{\frac{3}{2}}}.$ $\qquad \phi'(t)\,dt = \dfrac{3\,t^2dt}{(1 - t^2)^{\frac{5}{2}}}.$
14. $d\left[\dfrac{x\log x}{1 - x} + \log(1 - x)\right] = \dfrac{\log x\,dx}{(1 - x)^2}.$
15. $d\,[\text{arc tan} \log y] = \dfrac{dy}{y\,[1 + (\log y)^2]}.$
16. $d\left[r \text{ arc vers}\,\dfrac{y}{r} - \sqrt{2\,ry - y^2}\right] = \dfrac{ydy}{\sqrt{2\,ry - y^2}}.$
17. $d\left[\dfrac{\cos\phi}{2\sin^2\phi} - \dfrac{1}{2}\log\tan\dfrac{\phi}{2}\right] = -\,\dfrac{d\phi}{\sin^3\phi}.$

# CHAPTER XII

## RATES

**106. The derivative considered as the ratio of two rates.** Let

$$y = f(x)$$



be the equation of a curve generated by a moving point $P$. Its coördinates $x$ and $y$ may then be considered as functions of the time, as explained in § 84, p. 104. Differentiating with respect to $t$, by **XXVI**, we have

$$\textbf{(81)} \qquad \frac{dy}{dt} = f'(x)\frac{dx}{dt}.$$

*At any instant the time rate of change of y (or the function) equals its derivative multiplied by the time rate of change of the independent variable.*

Or, write (**81**) in the form

$$\textbf{(82)} \qquad \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = f'(x) = \frac{dy}{dx}.$$

*The derivative measures the ratio of the time rate of change of y to that of x.*

$\frac{ds}{dt}$ being the time rate of change of length of arc, we have from (**12**), p. 105,

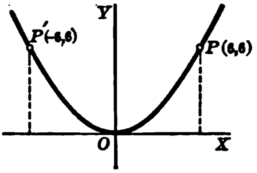
$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

which is the relation indicated by the above figure.

Ex. 1. A point moves on the parabola $6y = x^2$ in such a way that when $x = 6$ the abscissa is increasing at the rate of 2 ft. per second. At what rates are the ordinate and length of arc increasing at the same instant?

*Solution.* Differentiating with respect to the time $t$, we get

$$(A)\qquad \frac{dy}{dt}=\frac{1}{3}\cdot x\frac{dx}{dt}.$$



This means that at *any* point on the parabola (*Rate of change of* $y$) = ($\frac{1}{3}x$) (*Rate of change of* $x$).

But from the problem, when $x=6$, $\frac{dx}{dt}=2$ ft. per sec.

Substituting in ($A$), $\frac{dy}{dt}=\frac{1}{3}\cdot 6.2$ ft. per sec. $=4$ ft. per sec.

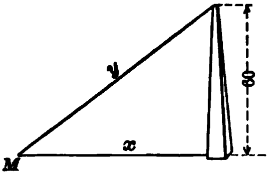
Also, $\frac{ds}{dt}=\sqrt{4+16}=2\sqrt{5}$ ft. per sec.

That is, at the point $P$ (6, 6) the ordinate changes in value twice as rapidly as the abscissa.

If we consider the point $P'(-6, 6)$ instead, the result is $\frac{dy}{dt}=-4$ ft. per sec.

The minus sign indicates that the ordinate is decreasing as the abscissa increases.

**Ex. 2.** A man is walking at the rate of 5 miles per hour towards the foot of a tower 60 ft. high standing on a horizontal plane. At what rate is he approaching the top (a) at any instant; (b) when he is 80 ft. from the foot of the tower?



*Solution.* Let $y$ = distance from top and $x$ = distance from foot of tower at any instant. Then $x^2+(60)^2=y^2$.

Differentiating with respect to the time $t$ gives

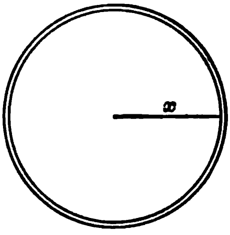
$$\frac{dy}{dt}=\frac{x}{y}\frac{dx}{dt}.$$

(a) Hence he is approaching the top $\frac{x}{y}$ times as fast as he is approaching the foot.

(b) When $x=80$, $y=100$; and we have given $\frac{dx}{dt}=5\times 5280$. Therefore

$$\frac{dy}{dt}=\frac{80}{100}\times 5\times 5280 \text{ ft. per hour}$$
$$=4 \text{ miles per hour.}$$

**Ex. 3.** A circular plate of metal expands by heat so that its diameter increases uniformly at the rate of .01 inch per minute. At what rate is the surface increasing (a) at any instant; (b) when the radius is 2 inches?



*Solution.* Let $x$ = radius and $y$ = area of the plate. Then $y=\pi x^2$, and differentiating with respect to the time $t$,

$$\frac{dy}{dt}=2\pi x\frac{dx}{dt}.$$

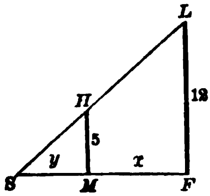
(a) That is, at any instant the area of the plate is increasing in square inches $2\pi x$ times as fast as the radius is increasing in linear inches.

(b) When $\frac{dx}{dt} = .01$ in. per sec. and $x = 2$ in.

$$\frac{dy}{dt} = .04\,\pi \text{ sq. in. per sec.};$$

i.e. the area is increasing .04 $\pi$ sq. in. per sec. at that instant.

Ex. 4. An arc light is hung 12 ft. directly above a straight horizontal walk on which a man 5 ft. in height is walking. How fast is the man's shadow lengthening when he is walking away from the light at the rate of 168 ft. per minute?



*Solution.* Let $x$ = distance of man from a point directly under light $L$, and $y$ = length of man's shadow. From the figure,

$$y : y + x :: 5 : 12,$$

or,

$$y = \tfrac{5}{7}x.$$

Differentiating,

$$\frac{dy}{dt} = \frac{5}{7}\frac{dx}{dt};$$

i.e. the shadow is lengthening $\frac{5}{7}$ as fast as the man is walking, or 120 ft. per minute.

## EXAMPLES

1. In a parabola $y^2 = 12x$, if $x$ increases uniformly at the rate of 2 in. per second, at what rate is $y$ increasing when $x = 3$ in.? *Ans.* 2 in. per sec.

2. At what point on the parabola of the last example do the abscissa and ordinate increase at the same rate? *Ans.* (3, 6).

3. In the function $y = 2x^3 + 6$, what is the value of $x$ at the point where $y$ increases 24 times as fast as $x$? *Ans.* $x = \pm 2$.

4. Find the value of $x$ when the function $2x^2 - 4$ is decreasing 5 times as rapidly as $x$ increases. *Ans.* $x = -\frac{5}{4}$.

5. Find the values of $x$ at the points where the rate of change of

$$x^3 - 12x^2 + 45x - 13$$

is zero. *Ans.* $x = 3$ and 5.

6. What is the value of $x$ at the point where $x^3 - 5x^2 + 17x$ and $x^3 - 3x$ change at the same rate? *Ans.* $x = 2$.

7. At what point on the ellipse $16x^2 + 9y^2 = 400$ does $y$ decrease at the same rate that $x$ increases? *Ans.* $(3, \frac{16}{3})$.

8. Given $y = x^3 - 6x^2 + 3x + 5$; find the points at which the rate of change of the ordinate is equal to the rate of change of the slope of the tangent to the curve. *Ans.* $x = 1$ and 5.

9. Where in the first quadrant does the arc increase twice as fast as the sine? *Ans.* At 60°.

10. The side of an equilateral triangle is 24 inches long, and is increasing at the rate of 3 inches per hour; how fast is the area increasing?

*Ans.* $36\sqrt{3}$ sq. in. per hour.

11. Find the rate of change of the area of a square when the side $b$ is increasing at the rate of $a$ units per second. *Ans.* $2\,ab$ sq. units per sec.

12. (a) The volume of a spherical soap bubble increases how many times as fast as the radius? (b) When its radius is 4 in. and increasing at the rate of $\frac{1}{2}$ in. per second, how fast is the volume increasing? *Ans.* (a) $4\,\pi r^2$ times as fast;
(b) $32\,\pi$ cu. in. per sec.

13. One end of a ladder 50 ft. long is leaning against a perpendicular wall standing on a horizontal plane. Supposing the foot of the ladder be pulled away from the wall at the rate of 3 ft. per minute; (a) how fast is the top of the ladder descending when the foot is 14 ft. from the wall? (b) when will the top and bottom of the ladder move at the same rate? (c) when is the top of the ladder descending at the rate of 4 ft. per minute? *Ans.* (a) $\frac{7}{8}$ ft. per sec.;
(b) when $25\sqrt{2}$ ft. from wall;
(c) when 40 ft. from wall.

14. A barge whose deck is 12 ft. below the level of a dock is drawn up to it by means of a cable attached to a ring in the floor of the dock, the cable being hauled in by a windlass on deck at the rate of 8 ft. per minute. How fast is the barge moving towards the dock when 16 ft. away? *Ans.* 10 ft. per minute.

15. An elevated car is 40 ft. immediately above a surface car, their tracks intersecting at right angles. If the speed of the elevated car is 16 miles per hour and of the surface car 8 miles per hour, at what rate are the cars separating 5 minutes after they met? *Ans.* 19.8 miles per hour.

16. One ship was sailing south at the rate of 6 miles per hour; another east at the rate of 8 miles per hour. At 4 P.M. the second crossed the track of the first where the first was two hours before. (a) How was the distance between the ships changing at 3 P.M.? (b) how at 5 P.M.? (c) when was the distance between them not changing? *Ans.* (a) Diminishing 2.8 miles per hour;
(b) increasing 8.73 miles per hour;
(c) 3.17 P.M.

17. Assuming the volume of the wood in a tree to be proportional to the cube of its diameter, and that the latter increases uniformly year by year when growing, show that the rate of growth when the diameter is 3 ft. is 36 times as great as when the diameter is 6 inches.

# CHAPTER XIII

## CHANGE OF VARIABLE

**107. Interchange of dependent and independent variables.** It is sometimes desirable to transform an expression involving derivatives of $y$ with respect to $x$ into an equivalent expression involving instead derivatives of $x$ with respect to $y$. Our examples will show that in many cases such a change transforms the given expression into a much simpler one. Or, perhaps $x$ is given as an explicit function of $y$ in a problem and it is found more convenient to use a formula involving $\frac{dx}{dy}, \frac{d^2x}{dy^2}$, etc., than one involving $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc. We shall now proceed to find the formulas necessary for making such transformations.

Given $y = f(x)$, then from **XXVII** we have

$$\textbf{(33)} \qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \qquad \frac{dy}{dx} \neq 0$$

giving $\frac{dy}{dx}$ in terms of $\frac{dx}{dy}$. Also, by **XXVI**,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\frac{dy}{dx}, \text{ or,}$$

$$(A) \qquad \frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}.$$

But $$\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right) = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}; \text{ and } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \text{ from } \textbf{(33)}.$$

Substituting these in $(A)$, we get

$$\textbf{(34)} \qquad \frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3},$$

giving $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$. Similarly,

$$(35)\qquad \frac{d^3y}{dx^3} = -\frac{\frac{d^3x}{dy^3}\frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5};$$

and so on for higher derivatives. This transformation is called *changing the independent variable from x to y.*

Ex. 1. Change the independent variable from $x$ to $y$ in the equation

$$3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

*Solution.* Substituting from (33), (34), (35),

$$3\left(-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\right)^2 - \left(\frac{1}{\frac{dx}{dy}}\right)\left(-\frac{\frac{d^3x}{dy^3}\frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5}\right) - \left(-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\right)\left(\frac{1}{\frac{dx}{dy}}\right)^2 = 0.$$

Reducing, we get

$$\frac{d^3x}{dy^3} + \frac{d^2x}{dy^2} = 0,$$

a much simpler equation.

### 108. Change of the dependent variable.

Let

$$(A)\qquad y = f(x),$$

and, suppose that at the same time $y$ is a function of $z$, say

$$(B)\qquad y = \phi(z).$$

We may then express $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc., in terms of $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, etc., as follows.

In general, $z$ is a function of $y$ by $(B)$, § 55, p. 57, and since $y$ is a function of $x$ by $(A)$, it is evident that $z$ is a function of $x$. Hence by **XXVI** we have

$$(C)\qquad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \phi'(z)\frac{dz}{dx}.$$

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\phi'(z)\frac{dz}{dx}\right) = \frac{dz}{dx}\frac{d}{dx}\phi'(z) + \phi'(z)\frac{d^2z}{dx^2}$. By **V**

But $\frac{d}{dx}\phi'(z) = \frac{d}{dz}\phi'(z)\frac{dz}{dx} = \phi''(z)\frac{dz}{dx}$. By **XXVI**

$$(D)\qquad \therefore\ \frac{d^2y}{dx^2} = \phi''(z)\left(\frac{dz}{dx}\right)^2 + \phi'(z)\frac{d^2z}{dx^2}.$$

Similarly for higher derivatives. This transformation is called *changing the dependent variable from $y$ to $z$*, the independent variable remaining $x$ throughout. We will now illustrate this process by means of an example.

**Ex. 1.** Having given the equation

$$(E) \qquad \frac{d^2y}{dx^2} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2,$$

change the dependent variable from $y$ to $z$ by means of the relation

$$(F) \qquad y = \tan z.$$

*Solution.* From $(F)$,

$$\frac{dy}{dx} = \sec^2 z\frac{dz}{dx},\ \frac{d^2y}{dx^2} = \sec^2 z\frac{d^2z}{dx^2} + 2\sec^2 z\tan z\left(\frac{dz}{dx}\right)^2.$$

Substituting in $(E)$,

$$\sec^2 z\frac{d^2z}{dx^2} + 2\sec z\tan z\left(\frac{dz}{dx}\right)^2 = 1 + \frac{2(1+\tan z)}{1+\tan z}\left(\sec^2 z\frac{dz}{dx}\right)^2,$$

and reducing, we get $\dfrac{d^2z}{dx^2} - 2\left(\dfrac{dz}{dx}\right)^2 = \cos^2 z.$ *Ans.*

**109. Change of the independent variable.** Let $y$ be a function of $x$, and at the same time let $x$ (and hence also $y$) be a function of a new variable $t$. It is required to express

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \text{etc.},$$

in terms of new derivatives having $t$ as the independent variable. By **XXVI**,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},\ \text{or,}$$

$$(A) \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

Also,
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}.$$

But differentiating $(A)$ with respect to $t$,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}\right) = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^2}.$$

Therefore

(*B*) $$\frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3};$$

and so on for higher derivatives. This transformation is called *changing the independent variable from x to t.* It is usually better to work out examples by the methods illustrated above rather than by using the formulas deduced.

Ex. 1. Change the independent variable from $x$ to $t$ in the equation

(*C*) $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0,$$

by means of the relation

(*D*) $$x=e^t.$$

*Solution.* $$\frac{dx}{dt}=e^t, \text{ therefore}$$

(*E*) $$\frac{dt}{dx}=e^{-t}.$$

Also, $$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}, \text{ therefore}$$

(*F*) $$\frac{dy}{dx}=e^{-t}\frac{dy}{dt}. \quad \text{Also,}$$

$$\frac{d^2y}{dx^2}=e^{-t}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}e^{-t}\frac{dt}{dx}=e^{-t}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}-\frac{dy}{dt}e^{-t}\frac{dt}{dx}.$$

Substituting in the last result from (*E*),

(*G*) $$\frac{d^2y}{dx^2}=e^{-2t}\frac{d^2y}{dt^2}-\frac{dy}{dt}e^{-2t}.$$

Substituting (*D*), (*F*), (*G*) in (*C*),

$$e^{2t}\left(e^{-2t}\frac{d^2y}{dt^2}-\frac{dy}{dt}e^{-2t}\right)+e^t\left(e^{-t}\frac{dy}{dt}\right)+y=0;$$

and reducing, we get $\frac{d^2y}{dt^2}+y=0.$ *Ans.*

Since the formulas deduced in the Differential Calculus generally involve derivatives of $y$ with respect to $x$, such formulas as (*A*) and (*B*) are especially useful when the parametric equations of a curve are given. Such examples were given on pp. 92–97, and many others will be employed in what follows.

**110. Simultaneous change of both independent and dependent variables.** It is often desirable to change both variables simultaneously. An important case is that arising in the transformation from rectangular to polar coördinates. Since

$$x = \rho\cos\theta \text{ and } y = \rho\sin\theta,$$

the equation

$$f(x, y) = 0$$

becomes by substitution an equation between $\rho$ and $\theta$ defining $\rho$ as a function of $\theta$. Hence $\rho$, $x$, $y$ are all functions of $\theta$.

Ex. 1. Transform the formula for the radius of curvature (42), p. 163,

$$(A)\qquad R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

into polar coördinates.

*Solution.* Since in $(A)$ and $(B)$, pp. 154, 155, $t$ is any variable on which $x$ and $y$ depend, we may in this case let $t = \theta$, giving

$$(B)\qquad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}, \text{ and}$$

$$(C)\qquad \frac{d^2y}{dx^2} = \frac{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3}.$$

Substituting $(B)$ and $(C)$ in $(A)$, we get

$$R = \left[\frac{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}{\left(\frac{dx}{d\theta}\right)^2}\right]^{\frac{3}{2}} \div \frac{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3}, \text{ or,}$$

$$(D)\qquad R = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}}.$$

But since $x = \rho\cos\theta$ and $y = \rho\sin\theta$, we have

$$\frac{dx}{d\theta} = -\rho\sin\theta + \cos\theta\frac{d\rho}{d\theta};\ \frac{dy}{d\theta} = \rho\cos\theta + \sin\theta\frac{d\rho}{d\theta};$$

$$\frac{d^2x}{d\theta^2} = -\rho\cos\theta - 2\sin\theta\frac{d\rho}{d\theta} + \cos\theta\frac{d^2\rho}{d\theta^2};\ \frac{d^2y}{d\theta^2} = -\rho\sin\theta + 2\cos\theta\frac{d\rho}{d\theta} + \sin\theta\frac{d^2\rho}{d\theta^2}.$$

**Substituting these in (*D*) and reducing,**

$$R=\frac{\left[\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2+2\left(\frac{d\rho}{d\theta}\right)^2-\rho\frac{d^2\rho}{d\theta^2}}.\quad \textit{Ans.}$$

## EXAMPLES

**Change the independent variable from $x$ to $y$ in the four following equations.**

1. $R=\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$ *Ans.* $R=-\dfrac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$

2. $\dfrac{d^2y}{dx^2}+2y\left(\dfrac{dy}{dx}\right)^2=0.$ *Ans.* $\dfrac{d^2x}{dy^2}-2y\dfrac{dx}{dy}=0.$

3. $x\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3-\dfrac{dy}{dx}=0.$ *Ans.* $x\dfrac{d^2x}{dy^2}-1+\left(\dfrac{dx}{dy}\right)^2=0.$

4. $\left(3a\dfrac{dy}{dx}+2\right)\left(\dfrac{d^2y}{dx^2}\right)^2=\left(a\dfrac{dy}{dx}+1\right)\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}.$ *Ans.* $\left(\dfrac{d^2x}{dy^2}\right)^2=\left(\dfrac{dx}{dy}+a\right)\dfrac{d^3x}{dy^3}.$

**Change the dependent variable from $y$ to $z$ in the following equation.**

5. $(1+y)^2\left(\dfrac{d^3y}{dx^3}-2y\right)+\left(\dfrac{dy}{dx}\right)^3=2\left(1+y\right)\dfrac{dy}{dx}\dfrac{d^2y}{dx^2},\quad y=z^2+2z.$

*Ans.* $(z+1)\dfrac{d^3z}{dx^3}=\dfrac{dz}{dx}\dfrac{d^2z}{dx^2}+z^2+2z.$

**Change the independent variable in the following eight equations.**

6. $\dfrac{d^2y}{dx^2}-\dfrac{x}{1-x^2}\dfrac{dy}{dx}+\dfrac{y}{1-x^2}=0,\quad x=\cos t.$ *Ans.* $\dfrac{d^2y}{dt^2}+y=0.$

7. $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=0,\quad x=\cos z.$ *Ans.* $\dfrac{d^2y}{dz^2}=0.$

8. $(1-y^2)\dfrac{d^2u}{dy^2}-y\dfrac{du}{dy}+a^2u=0,\quad y=\sin x.$ *Ans.* $\dfrac{d^2u}{dx^2}+a^2u=0.$

9. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+\dfrac{a^2}{x^2}y=0,\quad x=\dfrac{1}{z}.$ *Ans.* $\dfrac{d^2y}{dz^2}+a^2y=0.$

10. $x^3\dfrac{d^3v}{dx^3}+3x^2\dfrac{d^2v}{dx^2}+x\dfrac{dv}{dx}+v=0,\quad x=e^t.$ *Ans.* $\dfrac{d^3v}{dt^3}+v=0.$

11. $\dfrac{d^2y}{dx^2}+\dfrac{2x}{1+x^2}\dfrac{dy}{dx}+\dfrac{y}{(1+x^2)^2}=0,\quad x=\tan\theta.$ *Ans.* $\dfrac{d^2y}{d\theta^2}+y=0.$

12. $\dfrac{d^2u}{ds^2}+su\dfrac{du}{ds}+\operatorname{cosec}^2 s=0,\quad s=\arctan t.$

*Ans.* $(1+t^2)\dfrac{d^2u}{dt^2}+(2t+u\arctan t)\dfrac{du}{dx}+1=0.$

13. $x^4\dfrac{d^2y}{dx^2}+a^2y=0,\quad x=\dfrac{1}{z}.$ *Ans.* $\dfrac{d^2y}{dz^2}+\dfrac{2}{z}\dfrac{dy}{dz}+a^2y=0.$

In the following four examples the equations are given in parametric form. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each case.

14. $x = 7 + t^2,\ y = 3 + t^2 - 3t^4$. *Ans.* $\frac{dy}{dx} = 1 - 6t^2,\ \frac{d^2y}{dx^2} = -6$.

15. $x = \cot t,\ y = \sin^3 t$. *Ans.* $\frac{dy}{dx} = -3\sin^4 t \cos t,\ \frac{d^2y}{dx^2} = 3\sin^5 t(4 - 5\sin^2 t)$.

16. $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$.

*Ans.* $\frac{dy}{dx} = \tan t,\ \frac{d^2y}{dx^2} = \frac{1}{at\cos^3 t}$.

17. $x = \frac{1-t}{1+t},\ y = \frac{2t}{1+t}$. *Ans.* $\frac{dy}{dx} = -1,\ \frac{d^2y}{dx^2} = 0$.

18. Transform $\frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$ by assuming $x = \rho\cos\theta,\ y = \rho\sin\theta$.

*Ans.* $\frac{\rho^2}{\sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}}$.

19. Let $f(x, y) = 0$ be the equation of a curve. Find an expression for its slope $\left(\frac{dy}{dx}\right)$ in terms of polar coördinates.

*Ans.* $\frac{dy}{dx} = \frac{\rho\cos\theta + \sin\theta\frac{d\rho}{d\theta}}{-\rho\sin\theta + \cos\theta\frac{d\rho}{d\theta}}$.

# CHAPTER XIV

## CURVATURE. RADIUS OF CURVATURE

**111. Curvature.** The shape of a curve depends very largely upon the rate at which the direction of the tangent changes as the point of contact describes the curve. This rate of change of direction is called *curvature* and is denoted by $K$. We now proceed to find its analytical expression, first for the simple case of the circle, and then for curves in general.

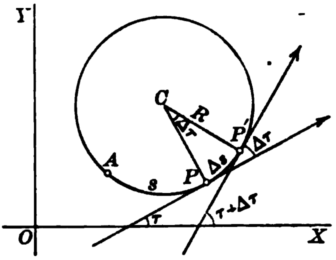
**112. Curvature of a circle.** Consider a circle of radius $R$. Let

$\tau =$ angle that the tangent at $P$ makes with $OX$, and

$\tau + \Delta\tau =$ angle made by the tangent at a neighboring point $P'$.

Then we say

$\Delta\tau =$ *total curvature of arc* $PP'$.



If the point $P$ with its tangent be supposed to move along the curve to $P'$, the total curvature ($= \Delta\tau$) would measure the total change in direction, or rotation, of the tangent; or, what is the same thing, the total change in direction of the arc itself. Denoting by $s$ the length of the arc of the curve measured from some fixed point (as $A$) to $P$, and by $\Delta s$ the length of the arc $PP'$, then the ratio

$$\frac{\Delta\tau}{\Delta s}$$

measures the average change in direction per unit length of arc.* Since from the figure,

$$\Delta s = R \cdot \Delta\tau,$$

or,

$$\frac{\Delta\tau}{\Delta s} = \frac{1}{R},$$

---

* Thus, if $\Delta\tau = \frac{\pi}{6}$ radians (= 30°), and $\Delta s = 3$ centimeters; then $\frac{\Delta\tau}{\Delta s} = \frac{\pi}{18}$ radians per centimeter = 10° per centimeter = average rate of change of direction.

it is evident that this ratio is constant everywhere on the circle. This ratio is by definition the *curvature of the circle*, and we have

$$K = \frac{1}{R}. \tag{36}$$

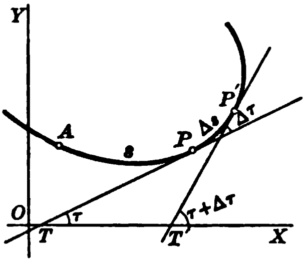
*The curvature of a circle equals the reciprocal of its radius.*

**113. Curvature at a point.** Consider any curve. As in the last section,

$$\Delta\tau = \textit{total curvature of the arc } PP',$$

and
$$\frac{\Delta\tau}{\Delta s} = \textit{average curvature of the arc } PP'.$$

More important, however, than the notion of the average curvature of an arc is that of *curvature at a point*. This is obtained as follows. Imagine $P'$ to approach $P$ along the curve, then the limiting value of the average curvature $\left(=\frac{\Delta\tau}{\Delta s}\right)$ as $P'$ approaches $P$ along the curve is defined as the curvature at $P$, i.e.



$$\textit{Curvature at a point} = \lim_{\Delta s = 0}\left(\frac{\Delta\tau}{\Delta s}\right) = \frac{d\tau}{ds}.$$

$$\therefore\ K = \frac{d\tau}{ds} = \textbf{curvature.} \tag{37}$$

If we suppose $P$ to move along the curve, $\tau$ and $s$ are functions of the time $t$, and we may write

$$K = \frac{d\tau}{ds} = \frac{\frac{d\tau}{dt}}{\frac{ds}{dt}},$$

where $\frac{d\tau}{dt}$ = angular velocity of rotation of the tangent,

and $\frac{ds}{dt} = v =$ magnitude of the velocity of $P$ in the path.

Let $v = 1$, then $K = \frac{d\tau}{dt}$,

and we have the theorem:

*The curvature at $P$ is equal to the angular velocity of the tangent at $P$ when $P$ describes the curve with unit velocity.*

**114. Formulas for curvature.** It is evident that if in the last section, instead of measuring the angles which the tangents made with $OX$, we had denoted by $\tau$ and $\tau + \Delta\tau$ the angles made by the tangents with any arbitrarily fixed line, the different steps would in no wise have been changed, and consequently the results are entirely independent of the system of coördinates used. However, since the equations of the curves we shall consider are all given in either rectangular or polar coördinates, it is necessary to deduce formulas for $K$ in terms of both. We have

$$\tan\tau = \frac{dy}{dx}, \qquad \text{§ 45, p. 44}$$

or,

$$\tau = \text{arc}\tan\frac{dy}{dx}.$$

Differentiating with respect to $s$,

$$\frac{d\tau}{ds} = \frac{\frac{d}{ds}\left(\frac{dy}{dx}\right)}{1 + \left(\frac{dy}{dx}\right)^2}, \qquad \text{by XXI}$$

$$(A) \qquad \frac{d\tau}{ds} = \frac{\frac{d^2y}{dx^2}\frac{dx}{ds}}{1 + \left(\frac{dy}{dx}\right)^2}. \qquad \text{By XXVI}$$

But

$$\frac{dx}{ds} = \frac{1}{\frac{ds}{dx}} = \frac{1}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}}. \qquad \text{By (34), p. 142}$$

Therefore, substituting in $(A)$,

$$\frac{d\tau}{ds} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}},$$

and, since $K = \frac{d\tau}{ds}$, we have

$$(38) \qquad K = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

If the equation of the curve be given in polar coördinates, $K$ may be found by transforming* the above formula by means of the relations

$$x = \rho \cos \theta, \quad y = \rho \sin \theta;$$

giving

$$K = \frac{\rho^2 - \rho \frac{d^2\rho}{d\theta^2} + 2\left(\frac{d\rho}{d\theta}\right)^2}{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}. \tag{39}$$

**Ex. 1.** Find the curvature of the parabola $y^2 = 4px$ at the upper end of the latus rectum.

*Solution.* $\frac{dy}{dx} = \frac{2p}{y}; \quad \frac{d^2y}{dx^2} = -\frac{2p}{y^2}\frac{dy}{dx} = -\frac{4p^2}{y^3}.$

Substituting in (38),

$$K = -\frac{4p^2}{(y^2 + 4p^2)^{\frac{3}{2}}};$$

giving the curvature at *any point*. At the upper end of the latus rectum $(p, 2p)$

$$K = -\frac{4p^2}{(4p^2 + 4p^2)^{\frac{3}{2}}} = -\frac{4p^2}{16\sqrt{2}p^3} = -\frac{1}{4\sqrt{2}p}.\dagger \quad \textit{Ans.}$$

**Ex. 2.** Find the curvature of the logarithmic spiral $\rho = e^{a\theta}$ at any point.

*Solution.* $\frac{d\rho}{d\theta} = ae^{a\theta} = a\rho; \quad \frac{d^2\rho}{d\theta^2} = a^2e^{a\theta} = a^2\rho.$

Substituting in (39),

$$K = \frac{1}{\rho\sqrt{1 + a^2}}. \quad \textit{Ans.}$$

**115. Radius of curvature.** By analogy with the circle [see (36) p. 160], the *radius of curvature of a curve at a point* is defined as the reciprocal of the curvature of the curve at that point. Denoting the radius of curvature by $R$, we have

$$R = \frac{1}{K};\ddagger$$

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* The details of this transformation were given in Ex. 1 on pp. 156, 157.

† While in our work it is generally only the numerical value of $K$ that is of importance, yet we can give a geometric meaning to its sign. Throughout our work we have taken the positive sign of the radical $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. Therefore $K$ will be positive or negative at the same time as $\frac{d^2y}{dx^2}$, that is (§ 98, p. 136), according as the curve is concave upwards or concave downwards.

‡ Hence the radius of curvature will have the same sign as the curvature, i.e. + or − according as the curve is concave upwards or concave downwards.

or, substituting the values of $K$ from (38) and (39),

$$(40)\qquad R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}};$$

$$(41)\qquad R = \frac{\left[\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2-\rho\frac{d^2\rho}{d\theta^2}+2\left(\frac{d\rho}{d\theta}\right)^2}.$$

**Ex. 1.** Find the radius of curvature at any point of the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$.

*Solution.* $\frac{dy}{dx}=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$; $\frac{d^2y}{dx^2}=\frac{1}{2a}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$. Substituting in (40),

$$R=\frac{\left[1+\left(\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2}\right)^2\right]^{\frac{3}{2}}}{\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2a}}=\frac{\left(\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2}\right)^3}{\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2a}}=\frac{a\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2}{4}=\frac{y^2}{a}.\quad \textit{Ans.}$$

If the equation of the curve is given in parametric form, find the first and second derivatives of $y$ with respect to $x$ from ($A$) and ($B$), p. 154, 155, namely:

$$(B)\qquad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}},\text{ and}$$

$$(C)\qquad \frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3};$$

and then substitute the results in (40).

**Ex. 2.** Find the radius of curvature of the cycloid

$$x=a(t-\sin t),$$
$$y=a(1-\cos t).$$

*Solution.* $\frac{dx}{dt}=a(1-\cos t)$, $\frac{dy}{dt}=a\sin t$;

$\frac{d^2x}{dt^2}=a\sin t$, $\frac{d^2y}{dt^2}=a\cos t$.

Substituting in (*B*) and (*C*), and then in (40), p. 163, we get

$$\frac{dy}{dx}=\frac{\sin t}{1-\cos t},\quad \frac{d^2y}{dx^2}=\frac{a(1-\cos t)a\cos t-a\sin t\, a\sin t}{a^3(1-\cos t)^3}=-\frac{1}{a(1-\cos t)^2}.$$

$$R=\frac{\left[1+\left(\frac{\sin t}{1-\cos t}\right)^2\right]^{\frac{3}{2}}}{-\frac{1}{a(1-\cos t)^2}}=-2a\sqrt{2-2\cos t}.\quad \textit{Ans.}$$

NOTE. From (6), p. 90, we get

$$\text{length of normal}=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=a(1-\cos t)\sqrt{1+\left(\frac{\sin t}{1-\cos t}\right)^2}$$
$$=a\sqrt{2-2\cos t}.$$

Hence from a comparison of the last two results:

*At any point on the cycloid the length of the radius of curvature is twice the length of the normal.*

### EXAMPLES

1. Find the radius of curvature of the equilateral hyperbola $xy=12$ at the point (3, 4). *Ans.* $R=\frac{125}{24}$.

2. What is the curvature of $y=x^4-4x^3-18x^2$ at the origin? *Ans.* $K=36$.

3. Find the radius of curvature of the ellipse $b^2x^2+a^2y^2=a^2b^2$ (a) at any point, (b) at end of major axis, (c) at end of minor axis $(a>b)$.

*Ans.* (a) $R=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$, (b) $R=\frac{b^2}{a}$, (c) $R=\frac{a^2}{b}$.

4. What is the radius of curvature of the curve $16y^2=4x^4-x^6$ (a) at (0, 0), (b) at (2, 0)? *Ans.* (a) $R=1$, (b) $R=2$.

5. Find the curvature of the cubical parabola $a^2y=x^3$.

*Ans.* $K=\frac{6a^4x}{(a^4+9x^4)^{\frac{3}{2}}}$.

6. Get the radius of curvature in the semicubical parabola $ay^2=x^3$.

*Ans.* $R=\frac{x^{\frac{1}{2}}(4a+9x)^{\frac{3}{2}}}{6a}$.

7. Find the radius of curvature of the curve $y=x^3+5x^2+6x$ at the origin.

*Ans.* $R=22.506$.

8. Find the point on the parabola $y^2=8x$ at which the radius of curvature is $7\frac{13}{16}$. *Ans.* $(\frac{9}{8}, 3)$.

9. Find the curvature of the curve $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ at the point $(0, b)$.

*Ans.* $K=\frac{3b}{a^2}$.

**10.** Determine the radius of curvature of the curve $a^2y = bx^2 + cx^2y$ at the origin. *Ans.* $R = \infty$.

**11.** Show that the radius of curvature of the witch $y^2 = \dfrac{a^2(a - x)}{x}$ at the vertex is $\dfrac{a}{2}$.

**12.** Find the radius of curvature of the curve $y = \log \sec x$ at any point.
*Ans.* $R = \sec x$.

**13.** Find $K$ at any point on the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $K = \dfrac{a^{\frac{1}{2}}}{2(x + y)^{\frac{3}{2}}}$.

**14.** Find $R$ at any point on the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $R = 3(axy)^{\frac{1}{3}}$.

**15.** Find $R$ at any point on the cycloid $x = r \operatorname{arc\,vers} \dfrac{y}{r} - \sqrt{2ry - y^2}$.
*Ans.* $R = 2\sqrt{2ry}$.

Find the radius of curvature of the following curves at any point.

**16.** The circle $\rho = a \sin \theta$. *Ans.* $R = \dfrac{a}{2}$.

**17.** The spiral of Archimedes $\rho = a\theta$. *Ans.* $R = \dfrac{(\rho^2 + a^2)^{\frac{3}{2}}}{\rho^2 + 2a^2}$.

**18.** The cardioid $\rho = a(1 - \cos \theta)$. *Ans.* $R = \frac{2}{3}\sqrt{2a\rho}$.

**19.** The lemniscate $\rho^2 = a^2 \cos 2\theta$. *Ans.* $R = \dfrac{a^2}{3\rho}$.

**20.** The parabola $\rho = a \sec^2 \dfrac{\theta}{2}$. *Ans.* $R = 2a \sec^3 \dfrac{\theta}{2}$.

**21.** The curve $\rho = a \sin^3 \dfrac{\theta}{3}$. *Ans.* $R = \frac{3}{4} a \sin^2 \dfrac{\theta}{3}$.

**22.** The trisectrix $\rho = 2a \cos \theta - a$. *Ans.* $R = \dfrac{a(5 - 4\cos \theta)^{\frac{3}{2}}}{9 - 6\cos \theta}$.

**23.** The equilateral hyperbola $\rho^2 \cos 2\theta = a^2$. *Ans.* $R = \dfrac{\rho^3}{a^2}$.

**24.** The conic $\rho = \dfrac{a(1 - e^2)}{1 - e\cos \theta}$. *Ans.* $R = \dfrac{a(1 - e^2)(1 - 2e\cos \theta + e^2)^{\frac{3}{2}}}{(1 - e\cos \theta)^3}$.

**25.** The curve $\begin{cases} x = 3t^2, \\ y = 3t - t^3. \end{cases}$ *Ans.* $R = \frac{3}{2}(1 + t^2)^2$.

**26.** The hypocycloid $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$ *Ans.* $R = 3a \sin t \cos t$.

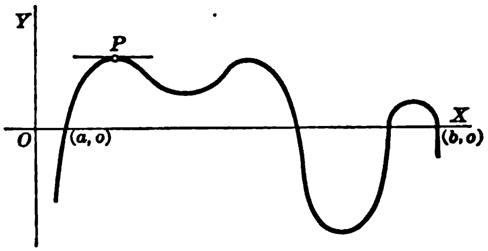
**27.** The curve $\begin{cases} x = a(\cos t + t \sin t), \\ y = a(\sin t - t \cos t). \end{cases}$ *Ans.* $R = at$.

**28.** The curve $\begin{cases} x = a(m \cos t + \cos mt), \\ y = a(m \sin t - \sin mt). \end{cases}$ *Ans.* $R = \dfrac{4ma}{m - 1} \sin\left(\dfrac{m + 1}{2}\right)t$.

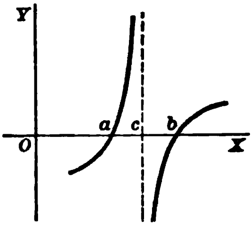
# CHAPTER XV

## THEOREM OF MEAN VALUE. INDETERMINATE FORMS

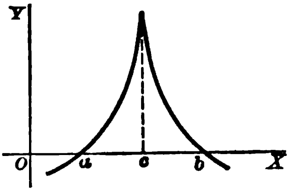
**116. Rolle's Theorem.** Let $y=f(x)$ be a continuous single-valued function of $x$ vanishing for $x=a$ and $x=b$, and suppose that $f'(x)$ changes continuously when $x$ varies from $a$ to $b$. The function will then be represented graphically by a continuous curve with a continuously turning tangent as in the figure. Geometric intuition shows us at once that for *at least one value of $x$ between $a$ and $b$* the tangent is parallel to the axis of $X$ (as at $P$), i.e. the slope $[=f'(x)]$ is zero. This illustrates **Rolle's Theorem:**



*If $f(x)$ vanishes when $x=a$ and $x=b$, and $f(x)$ and $f'(x)$ are continuous for all values of $x$ from $x=a$ to $x=b$, then $f'(x)$ will be zero for at least one value of $x$ between $a$ and $b$.**



This theorem is obviously true, because as $x$ increases from $a$ to $b$, $f(x)$ cannot always increase or always decrease as $x$ increases, since $f(a)=0$ and $f(b)=0$. Hence for at least one value of $x$ between $a$ and $b$, $f(x)$ must cease to increase and begin to decrease, or else cease to decrease and begin to increase; and for that particular value of $x$ the first derivative must be zero (§ 93, p. 118).



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* The second figure shows the graph of a function which is discontinuous ($=\infty$) for $x=c$, a value lying between $a$ and $b$. The third figure shows the graph of a continuous function whose first derivative does not exist for such an intermediate value $x=c$. In each case it is seen that at no point on the graph between $x=a$ and $x=b$ does the tangent (or curve) become parallel to $OX$.

**117. The Theorem of Mean Value.*** Consider the quantity $Q$ defined by the equation

$$(A) \qquad \frac{f(b)-f(a)}{b-a} = Q, \text{ or,}$$

$$(B) \qquad f(b)-f(a)-(b-a)\,Q = 0.$$

Let $F(x)$ be a function formed by replacing $b$ by $x$ in the left-hand member of $(B)$; that is,

$$(C) \qquad F(x) = f(x) - f(a) - (x-a)\,Q.$$

From $(B)$, $F(b) = 0$, and from $(C)$, $F(a) = 0$,

therefore by Rolle's Theorem, p. 166, $F'(x)$ † must be zero for at least one value of $x$ *between* $a$ and $b$, say $x_1$. But by differentiating $(C)$ we get

$$F'(x) = f'(x) - Q.$$

Therefore, since $F'(x_1) = 0$, then also $f'(x_1) - Q = 0$, and

$$Q = f'(x_1).$$

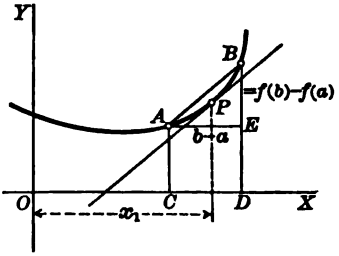
Substituting this value of $Q$ in $(A)$, we get the **Theorem of Mean Value,**

$$(42) \qquad \frac{f(b)-f(a)}{b-a} = f'(x_1), \qquad a < x_1 < b$$

where in general all we know about $x_1$ is that it lies *between* $a$ and $b$.

The Theorem of Mean Value can be easily interpreted geometrically. Let the curve in the figure be the locus of

$$y = f(x).$$



Take $OC = a$ and $OD = b$; then $f(a) = CA$ and $f(b) = DB$, giving $AE = b - a$ and $EB = f(b) - f(a)$.

Therefore the slope of the chord $AB$ is

$$\frac{EB}{AE} = \frac{f(b)-f(a)}{b-a},$$

and the Theorem of Mean Value simply asserts that there is at least one point on the curve between $A$ and $B$ (as $P$) where the tangent (or curve) is parallel to the chord $AB$. If the abscissa of $P$ is $x_1$, the slope at $P$ is $f'(x_1)$, and we have

$$\frac{f(b)-f(a)}{b-a} = f'(x_1).$$

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* Also called the *Law of the Mean*.

† If $F(x)$ and $F'(x)$ are continuous.

The student should draw curves to show that there may be more than one such point in the interval, and curves to illustrate on the other hand that the theorem may not be true if $f(x)$ becomes discontinuous for any value of $x$ between $a$ and $b$, or if $f'(x)$ ceases to exist.

Clearing (42) of fractions, we may also write the theorem in the form

$$f(b) = f(a) + (b - a)f'(x_1). \tag{43}$$

Let $b = a + \Delta a$; then $b - a = \Delta a$, and since $x_1$ is a number lying between $a$ and $b$, we may write

$$x_1 = a + \theta \cdot \Delta a,$$

where $\theta$ is a positive proper fraction. Substituting in (43), we get *another form of the* **Theorem of Mean Value,**

$$f(a + \Delta a) - f(a) = \Delta a f'(a + \theta \cdot \Delta a). \qquad 0 < \theta < 1 \tag{44}$$

**118. The Extended Theorem of Mean Value.*** Following the method of the last section, let $R$ be defined by the equation

$$f(b) - f(a) - (b - a)f'(a) - \tfrac{1}{2}(b - a)^2 R = 0. \tag{A}$$

Let $F(x)$ be a function formed by replacing $b$ by $x$ in the left-hand member of $(A)$; that is,

$$F(x) = f(x) - f(a) - (x - a)f'(a) - \tfrac{1}{2}(x - a)^2 R. \tag{B}$$

From $(A)$, $F(b) = 0$; and from $(B)$, $F(a) = 0$, therefore, by Rolle's Theorem, p. 166, at least one value of $x$ between $a$ and $b$, say $x_1$, will cause $F'(x)$ to vanish. Hence, since

$$F'(x) = f'(x) - f'(a) - (x - a)R, \text{ we get}$$
$$F'(x_1) = f'(x_1) - f'(a) - (x_1 - a)R = 0.$$

Since $F'(x_1) = 0$ and $F'(a) = 0$, it is evident that $F'(x)$ also satisfies the conditions of Rolle's Theorem, so that *its derivative*, namely $F''(x)$, must vanish for at least one value of $x$ between $a$ and $x_1$, say $x_2$, and therefore $x_2$ also lies between $a$ and $b$. But

$$F''(x) = f''(x) - R; \text{ therefore } F''(x_2) = f''(x_2) - R = 0, \text{ and}$$
$$R = f''(x_2).$$

Substituting this result in $(A)$, we get

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{\underline{|2}}(b - a)^2 f''(x_2). \qquad a < x_2 < b \tag{C}$$

---

* Also called the *Extended Law of the Mean.*

In the same manner, if we define $S$ by means of the equation

$$f(b)-f(a)-(b-a)f'(a)+\frac{1}{\underline{|2}}(b-a)^2f''(a)+\frac{1}{\underline{|3}}(b-a)^3S=0,$$

we can derive the equation

$$(D)\quad f(b)=f(a)-(b-a)f'(a)+\frac{1}{\underline{|2}}(b-a)^2f''(a)+\frac{1}{\underline{|3}}(b-a)^3f'''(x_3),\qquad a<x_3<b$$

where $x_3$ lies between $a$ and $b$.

By continuing this process we get the general result,*

$$(E)\quad f(b)=f(a)+\frac{(b-a)}{\underline{|1}}f'(a)+\frac{(b-a)^2}{\underline{|2}}f''(a)+\frac{(b-a)^3}{\underline{|3}}f'''(a)+\cdots$$
$$+\frac{(b-a)^{n-1}}{\underline{|n-1}}f^{(n-1)}(a)+\frac{(b-a)^n}{\underline{|n}}f^{(n)}(x_1),\qquad a<x_1<b$$

where $x_1$ lies between $a$ and $b$. $(E)$ is called the *Extended Theorem of Mean Value.*

**119. Maxima and minima treated analytically.** By making use of the results of the last two sections we can now give a general discussion of *maxima and minima of functions of a single independent variable.*

Given the function $f(x)$. Let $h$ be a positive number as small as we please; then the definitions given in § 94, p. 118, may be stated as follows:

If, for all values of $x$ different from $a$ in the interval $[a-h, a+h]$,

$$(A)\qquad f(x)-f(a)=\textit{a negative number},$$

then $f(x)$ is said to be a *maximum when* $x=a$.

If, on the other hand,

$$(B)\qquad f(x)-f(a)=\textit{a positive number},$$

then $f(x)$ is said to be a *minimum when* $x=a$.

Consider the following cases:

I. *Let* $f'(a)\neq 0$.

From **(48)**, p. 168, replacing $b$ by $x$ and transposing $f(a)$,

$$(C)\qquad f(x)-f(a)=(x-a)f'(x_1).\qquad a<x_1<x$$

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* It is assumed that $f(x)$, $f'(x)$, $f''(x)$, $\cdots$, $f^{(n)}(x)$ exist throughout the interval $[a, b]$.

Since $f'(a) \neq 0$, and $f'(x)$ is assumed as continuous, $h$ may be chosen so small that $f'(x)$ will have the same sign as $f'(a)$ for all values of $x$ in the interval $[a-h, a+h]$. Therefore $f'(x_1)$ has the same sign as $f'(a)$ (§§ 29–33). But $x-a$ changes sign according as $x$ is less or greater than $a$. Therefore, from $(C)$, the difference

$$f(x)-f(a)$$

will also change sign, and by $(A)$ and $(B)$, $f(a)$ will be neither a maximum nor a minimum. This result agrees with the discussion in § 94, where it was shown that *for all values of $x$ for which $f(x)$ is a maximum or a minimum the first derivative $f'(x)$ must vanish.*

II. *Let $f'(a)=0$, and $f''(a) \neq 0$.*

From $(C)$, p. 168, replacing $b$ by $x$ and transposing $f(a)$,

$$(D) \qquad f(x)-f(a)=\frac{(x-a)^2}{\underline{|2}}f''(x_2). \qquad a<x_2<x$$

Since $f''(a) \neq 0$, and $f''(x)$ is assumed as continuous, we may choose our interval $[a-h, a+h]$ so small that $f''(x_2)$ will have the same sign as $f''(a)$ (§§ 29–33). Also $(x-a)^2$ does not change sign. Therefore the second member of $(D)$ will not change sign, and the difference

$$f(x)-f(a)$$

will have the same sign for all values of $x$ in the interval $[a-h, a+h]$, and moreover *this sign will be the same as the sign of $f''(a)$.* It therefore follows from our definitions $(A)$ and $(B)$ that

$(E)$ *$f(a)$ is a maximum if $f'(a)=0$ and $f''(a)=$ a negative number;*

$(F)$ *$f(a)$ is a minimum if $f'(a)=0$ and $f''(a)=$ a positive number.*

These conditions are the same as (21) and (22), p. 124.

III. *Let $f'(a)=f''(a)=0$, and $f'''(a) \neq 0$.*

From $(D)$, p. 169, replacing $b$ by $x$ and transposing $f(a)$,

$$(G) \qquad f(x)-f(a)=\frac{1}{\underline{|3}}(x-a)^3 f'''(x_3). \qquad a<x_3<x$$

As before, $f'''(x_3)$ will have the same sign as $f'''(a)$. But $(x-a)^3$ changes its sign from $-$ to $+$ as $x$ increases through $a$. Therefore the difference

$$f(x)-f(a)$$

must change sign, and $f(a)$ is neither a maximum nor a minimum.

IV. *Let* $f'(a) = f''(a) = \cdots\cdots = f^{(n-1)}(a) = 0$, *and* $f^{(n)}(a) \neq 0$.

By continuing the process as illustrated in I, II, and III, it is seen that if the first derivative of $f(x)$ which does not vanish for $x = a$ is of even order ($= n$), then

**(45)** **$f(a)$ is a maximum if $f^{(n)}(a) =$ a negative number;**

**(46)** **$f(a)$ is a minimum if $f^{(n)}(a) =$ a positive number.**[*]

If the first derivative of $f(x)$ which does not vanish for $x = a$ is of **odd order, then $f(a)$ will be neither a maximum nor a minimum.**

Ex. 1. Examine $x^3 - 9x^2 + 24x - 7$ for maximum and minimum values.

*Solution.*
$$f(x) = x^3 - 9x^2 + 24x - 7.$$
$$f'(x) = 3x^2 - 18x + 24.$$

Solving
$$3x^2 - 18x + 24 = 0$$
gives the critical values $x = 2$ and $x = 4$. $\therefore f(2) = 0$, and $f(4) = 0$.

Differentiating again,
$$f''(x) = 6x - 18.$$

Since $f''(2) = -6$, we know from (45) that $f(2) = 13$ is a maximum.
Since $f''(4) = +6$, we know from (46) that $f(4) = 9$ is a minimum.

Ex. 2. Examine $e^x + 2\cos x + e^{-x}$ for maximum and minimum values.

*Solution.*
$$f(x) = e^x + 2\cos x + e^{-x},$$
$$f'(x) = e^x - 2\sin x - e^{-x} = 0, \text{ for } x = 0,\dagger$$
$$f''(x) = e^x - 2\cos x + e^{-x} = 0, \text{ for } x = 0,$$
$$f'''(x) = e^x + 2\sin x - e^{-x} = 0, \text{ for } x = 0,$$
$$f^{\text{iv}}(x) = e^x + 2\cos x + e^{-x} = 4, \text{ for } x = 0.$$

Hence, from (46), $f(0) = 4$ is a minimum.

### EXAMPLES

Examine the following functions for maximum and minimum values, using method of the last section.

1. $3x^4 - 4x^3 + 1$. *Ans.* $x = 1$ gives min. $= 0$;
$x = 0$ gives neither.

2. $x^3 - 6x^2 + 12x + 48$. *Ans.* $x = 2$ gives neither.

3. $(x-1)^2(x+1)^3$. *Ans.* $x = 1$ gives min. $= 0$;
$x = \frac{1}{5}$ gives max.;
$x = -1$ gives neither.

4. Investigate $x^5 - 5x^4 + 5x^3 - 1$, at $x = 1$ and $x = 3$.

5. Investigate $x^3 - 3x^2 + 3x + 7$, at $x = 1$.

6. Show that if the first derivative of $f(x)$ which does not vanish for $x = a$ is of odd order ($= n$), then $f(x)$ is an increasing or decreasing function when $x = a$, according as $f^{(n)}(a)$ is positive or negative.

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[*] As in § 94, a critical value $x = a$ is found by placing the first derivative equal to zero and solving the resulting equation for real roots.

† $x = 0$ is the only root of the equation $e^x - 2\sin x - e^{-x} = 0$.

**120. The Generalized Theorem of Mean Value.** This theorem asserts about the two functions $f(x)$ and $F(x)$ that

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(x_1)}{F'(x_1)}, \qquad a < x_1 < b \tag{47}$$

where $x_1$ lies in the interval $[a, b]$ and $F'(x)$ does not vanish in the interval.

*Proof.* By multiplying both sides of (47) by $F'(x_1)$ and transposing $f'(x_1)$ to the left-hand side, we see that this theorem requires that the equation

$$\frac{f(b)-f(a)}{F(b)-F(a)}F'(x)-f'(x)=0$$

shall have a root $x_1$ lying between $a$ and $b$. In order to make it possible to apply Rolle's Theorem, p. 166, let us try to construct a function having this left-hand member for a derivative and such that it (the function) vanishes for $x=a$ and $x=b$. Such a function is

$$\frac{f(b)-f(a)}{F(b)-F(a)}[F(x)-F(a)]-[f(x)-f(a)],$$

because it vanishes for $x=a$ and $x=b$, and hence by Rolle's Theorem its derivative must vanish for an intermediate value of $x$, say $x_1$; that is,

$$\frac{f(b)-f(a)}{F(b)-F(a)}F'(x_1)-f'(x_1)=0,$$

which is equivalent to (47).

**121. Indeterminate forms.** When, for a particular value of the independent variable, a function takes on one of the forms

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty,$$

it is said to be *indeterminate*, and the function is *not* defined for that value of the independent variable by the given analytical expression. For example, suppose we have

$$y=\frac{f(x)}{F(x)},$$

where for some value of the variable, as $x=a$,

$$f(a)=0,\ F(a)=0.$$

For this value of $x$ our function is *not* defined and we may therefore assign to it any value we please. It is evident from what has gone before (Case II, p. 23) that it is desirable to assign to the function a value that will make it continuous when $x = a$, whenever it is possible to do so.

**122. Evaluation of a function taking on an indeterminate form.** If when $x = a$ the function $f(x)$ assumes an indeterminate form, then

$$\underset{x=a}{\text{limit}} f(x)^*$$

*is taken as the value of $f(x)$ for $x = a$.*

The assumption of this limiting value makes $f(x)$ continuous for $x = a$. This agrees with the theorem under Case II, p. 23, and also with our practice in Chapter IV, where several functions assuming the indeterminate form $\frac{0}{0}$ were evaluated. Thus, for $x = 2$, the function $\frac{x^2-4}{x-2}$ assumes the form $\frac{0}{0}$, but

$$\underset{x=2}{\text{limit}} \frac{x^2-4}{x-2} = 4.$$

Hence 4 is taken as the value of the function for $x = 2$. Let us now illustrate graphically the fact that if we assume 4 as the value of the function for $x = 2$, then the function is continuous for $x = 2$.

Let $$y = \frac{x^2-4}{x-2}.$$

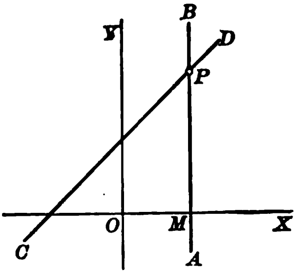
This equation may also be written in the form

$$y(x-2) = (x-2)(x+2),$$

or, $$(x-2)(y-x-2) = 0.$$

Placing each factor separately equal to zero, we have

$$x = 2, \text{ and } y = x + 2.$$



In plotting, the loci of these equations are found to be the two lines $AB$ and $CD$ respectively. Since there are infinitely many points on the line $AB$ having the abscissa 2, it is clear that when $x = 2$ $(= OM)$, the value of $y$ (or the function) may be taken as any

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[*] The calculation of this limiting value is called *evaluating the indeterminate form.*

number whatever, but when $x$ is different from 2 it is seen from the graph of the function that the corresponding value of $y$ (or the function) is always found from

$$y = x + 2,$$

the equation of the line $CD$. Also, on $CD$, when $x = 2$, we get

$$y = MP = 4,$$

which we saw was also the limiting value of $y$ (or the function) for $x = 2$; and it is evident from geometrical considerations that if we assume 4 as the value of the function for $x = 2$, then the function is continuous for $x = 2$.

Similarly, several of the examples given in Chapter IV illustrate how the limiting values of many functions assuming indeterminate forms may be found by employing suitable algebraic or trigonometric transformations, and how in general these limiting values make the corresponding functions continuous at the points in question. The most general methods, however, for evaluating indeterminate forms depend on differentiation.

**123. Evaluation of the indeterminate form $\frac{0}{0}$.** Given a function of the form $\frac{f(x)}{F(x)}$ such that $f(a) = 0$ and $F(a) = 0$; that is, the function takes on the indeterminate form $\frac{0}{0}$ when $a$ is substituted for $x$. It is then required to find

$$\underset{x=a}{\text{limit}} \frac{f(x)}{F(x)}.$$

Considering the functions $f(x)$ and $F(x)$ the same as in § 120 and replacing $b$ by $x$ in (47), p. 172, we get

$$\frac{f(x)}{F(x)} = \frac{f'(x_1)}{F'(x_1)}.$$

[Since $f(a) = 0$, and $F(a) = 0$.]

Since $x_1$ lies between $x$ and $a$, $x_1$ approaches $a$ as its limit when $x$ approaches $a$, and we have

$$\underset{x=a}{\text{limit}} \frac{f(x)}{F(x)} = \underset{x_1=a}{\text{limit}} \frac{f'(x_1)}{F'(x_1)} = \underset{x=a}{\text{limit}} \frac{f'(x)}{F'(x)}.\text{*}$$

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* Assuming that $\frac{f'(x)}{F'(x)}$ does approach a limit as $x$ approaches $a$.

If $f'(x)$ divided by $F'(x)$ does not assume an indeterminate form for $x = a$, we may write

$$\underset{x=a}{\text{limit}} \frac{f(x)}{F(x)} = \frac{f'(a)}{F'(a)}, \tag{48}$$

where $f(a) = 0$, $F(a) = 0$, $F'(a) \neq 0$. Hence

**Rule for evaluating the indeterminate form $\frac{0}{0}$.** *Differentiate the numerator for a new numerator and the denominator for a new denominator.** *The value of this new fraction for the assigned value*† *of the variable will be the limiting value of the original fraction.*

In case it so happens that

$$f'(a) = 0 \text{ and } F'(a) = 0,$$

that is, the first derivatives also vanish for $x = a$, then we still have the indeterminate form

$$\frac{0}{0},$$

and the theorem can be applied anew to the ratio

$$\frac{f'(x)}{F'(x)},$$

giving us

$$\underset{x=a}{\text{limit}} \frac{f(x)}{F(x)} = \frac{f''(a)}{F''(a)}.$$

When also $f''(a) = 0$ and $F''(a) = 0$, we get in the same manner

$$\underset{x=a}{\text{limit}} \frac{f(x)}{F(x)} = \frac{f'''(a)}{F'''(a)},$$

and so on.

It may be necessary to repeat this process several times.

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* The student is warned against the very careless but common mistake of differentiating the whole expression as a fraction by VIII.

† If $a = \infty$, the substitution $x = \frac{1}{z}$ reduces the problem to the evaluation of the limit for $z = 0$;

thus,
$$\underset{x=\infty}{\text{limit}} \frac{f(x)}{F(x)} = \underset{z=0}{\text{limit}} \frac{-f'\left(\frac{1}{z}\right)\frac{1}{z^2}}{-F'\left(\frac{1}{z}\right)\frac{1}{z^2}} = \underset{z=0}{\text{limit}} \frac{f'\left(\frac{1}{z}\right)}{F'\left(\frac{1}{z}\right)} = \underset{x=\infty}{\text{limit}} \frac{f'(x)}{F'(x)}.$$

Therefore the rule holds in this case also.

**Ex. 1.** Evaluate $\frac{f(x)}{F(x)} = \frac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}$ when $x = 1$.

*Solution.* $$\frac{f(1)}{F(1)} = \left.\frac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}\right]_{x=1} = \frac{1 - 3 + 2}{1 - 1 - 1 + 1} = \frac{0}{0}. \quad \therefore \textbf{ indeterminate.}$$

$$\frac{f'(1)}{F'(1)} = \left.\frac{3x^2 - 3}{3x^2 - 2x - 1}\right]_{x=1} = \frac{3 - 3}{3 - 2 - 1} = \frac{0}{0}. \quad \therefore \textbf{ indeterminate.}$$

$$\frac{f''(1)}{F''(1)} = \left.\frac{6x}{6x - 2}\right]_{x=1} = \frac{6}{6 - 2} = \frac{3}{2}. \quad \textit{Ans.}$$

**Ex. 2.** Evaluate $\underset{x=0}{\text{limit}} \frac{e^x + e^{-x} - 2x}{x - \sin x}$.

*Solution.* $$\frac{f(0)}{F(0)} = \left.\frac{e^x - e^{-x} - 2x}{x - \sin x}\right]_{x=0} = \frac{1 - 1 - 0}{0 - 0} = \frac{0}{0}. \quad \therefore \textbf{ indeterminate.}$$

$$\frac{f'(0)}{F'(0)} = \left.\frac{e^x + e^{-x} - 2}{1 - \cos x}\right]_{x=0} = \frac{1 + 1 - 2}{1 - 1} = \frac{0}{0}. \quad \therefore \textbf{ indeterminate.}$$

$$\frac{f''(0)}{F''(0)} = \left.\frac{e^x - e^{-x}}{\sin x}\right]_{x=0} = \frac{1 - 1}{0} = \frac{0}{0}. \quad \therefore \textbf{ indeterminate.}$$

$$\frac{f'''(0)}{F'''(0)} = \left.\frac{e^x + e^{-x}}{\cos x}\right]_{x=0} = \frac{1 + 1}{1} = 2. \quad \textit{Ans.}$$

## EXAMPLES

**Evaluate the following by differentiation.***

1. $\underset{x=4}{\text{limit}} \frac{x^2 - 16}{x^2 + x - 2}$. *Ans.* $\frac{8}{9}$.

2. $\underset{x=1}{\text{limit}} \frac{x - 1}{x^n - 1}$. $\frac{1}{n}$.

3. $\underset{x=1}{\text{limit}} \frac{\log x}{x - 1}$. 1.

4. $\underset{x=0}{\text{limit}} \frac{e^x - e^{-x}}{\sin x}$. 2.

5. $\underset{x=0}{\text{limit}} \frac{\tan x - x}{x - \sin x}$. 2.

6. $\underset{x=\frac{\pi}{2}}{\text{limit}} \frac{\log \sin x}{(\pi - 2x)^2}$. $-\frac{1}{8}$.

7. $\underset{x=0}{\text{limit}} \frac{a^x - b^x}{x}$. $\log \frac{a}{b}$.

8. $\underset{r=a}{\text{limit}} \frac{r^3 - ar^2 - a^2 r + a^3}{r^2 - a^2}$. 0.

9. $\underset{\theta=0}{\text{limit}} \frac{\theta - \text{arc} \sin \theta}{\sin^3 \theta}$. *Ans.* $-\frac{1}{6}$.

10. $\underset{x=\phi}{\text{limit}} \frac{\sin x - \sin \phi}{x - \phi}$. $\cos \phi$.

11. $\underset{y=0}{\text{limit}} \frac{e^y + \sin y - 1}{\log(1 + y)}$. 2.

12. $\underset{\theta=0}{\text{limit}} \frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1}$. 1.

13. $\underset{\phi=\frac{\pi}{4}}{\text{limit}} \frac{\sec^2 \phi - 2 \tan \phi}{1 + \cos' 4\phi}$. $\frac{1}{2}$.

14. $\underset{z=a}{\text{limit}} \frac{az - z^2}{a^4 - 2a^3 z + 2az^3 - z^4}$. $-\infty$.

15. $\underset{x=2}{\text{limit}} \frac{(e^x - e^2)^3}{(x - 4)e^x + e^2 x}$. $6e^4$.

---

* After differentiating, the student should in every case reduce the resulting expression to its simplest possible form before substituting the value of the variable.

**124. Evaluation of the indeterminate form $\frac{\infty}{\infty}$.**

In order to find
$$\underset{x=a}{\text{limit}}\frac{f(x)}{F(x)}$$
when
$$\underset{x=a}{\text{limit}} f(x)=\infty \text{ and } \underset{x=a}{\text{limit}} F(x)=\infty,$$
that is, when for $x=a$ the function
$$\frac{f(x)}{F(x)}$$
assumes the indeterminate form
$$\frac{\infty}{\infty},$$
we follow the same rule as that given on p. 175 for evaluating the indeterminate form $\frac{0}{0}$. Hence

**Rule for evaluating the indeterminate form $\frac{\infty}{\infty}$.** *Differentiate the numerator for a new numerator and the denominator for a new denominator. The value of this new fraction for the assigned value of the variable will be the limiting value of the original fraction.*[*]

In case the new fraction is indeterminate for the given value of the variable, we repeat the process as in the last section.

To prove this rule we must show that
$$\underset{x=a}{\text{limit}}\frac{f(x)}{F(x)}=\frac{f'(a)}{F'(a)}$$
when
$$\underset{x=a}{\text{limit}} f(x)=\infty,\ \underset{x=a}{\text{limit}} F(x)=\infty.$$

First we assume that
$$\underset{x=a}{\text{limit}}\frac{f'(x)}{F'(x)}=l, \text{ where } l \text{ is a definite number.}$$

From the Generalized Theorem of Mean Value, p. 172, we have
$$(A)\qquad \frac{f(x)-f(b)}{F(x)-F(b)}=\frac{f'(x_1)}{F'(x_1)},\qquad a<x<x_1<b<a+h$$
[Replacing $b$ and $a$ by $x$ and $b$ respectively.]

where $b$ is an arbitrary number and $F'(x)$ does not vanish in the interval $[a, a+h]$. From $(A)$
$$f(x)-f(b)=\frac{f'(x_1)}{F'(x_1)}[F(x)-F(b)],$$
or,
$$f(x)=f(b)+\frac{f'(x_1)}{F'(x_1)}[F(x)-F(b)].$$

---

[*] $f'(x)$ and $F'(x)$ are assumed to be continuous.

Dividing through by $F(x)$,

$$(B) \qquad \frac{f(x)}{F(x)} = \frac{f(b)}{F(x)} + \frac{f'(x_1)}{F'(x_1)}\left[1 - \frac{F(b)}{F(x)}\right].$$

By making $h$ sufficiently small it is evident that the difference

$$\frac{f'(x)}{F'(x)} - \frac{f'(x_1)}{F'(x_1)}$$

may be made as small as we please. Hence if we let $x$ approach the limit $a$ we get from $(B)$

$$\underset{x=a}{\text{limit}} \frac{f(x)}{F(x)} = \underset{x=a}{\text{limit}} \frac{f'(x)}{F'(x)} = \frac{f'(a)}{F'(a)}.$$

$$\left[\text{Since } \underset{x=a}{\text{limit}} \frac{f(b)}{F(x)} = 0,\ \underset{x=a}{\text{limit}} \frac{F(b)}{F(x)} = 0.\right]$$

In the same manner the rule may be shown to hold true when

$$\underset{x=a}{\text{limit}} \frac{f'(x)}{F'(x)} = \infty.$$

**Ex. 1.** Evaluate $\frac{\log x}{\csc x}$ for $x = 0$.

*Solution.* $\frac{f(0)}{F(0)} = \left.\frac{\log x}{\csc x}\right]_{x=0} = \frac{-\infty}{\infty}$. $\therefore$ indeterminate.

$$\frac{f'(0)}{F'(0)} = \left.\frac{\frac{1}{x}}{-\csc x \cot x}\right]_{x=0} = -\left.\frac{\sin^2 x}{x \cos x}\right]_{x=0} = \frac{0}{0}. \quad \therefore \text{ indeterminate.}$$

$$\frac{f''(0)}{F''(0)} = -\left.\frac{2 \sin x \cos x}{\cos x - x \sin x}\right]_{x=0} = -\frac{0}{1} = 0. \quad \textit{Ans.}$$

**125. Evaluation of the indeterminate form $0 \cdot \infty$.** If a function $f(x) \cdot \phi(x)$ takes on the indeterminate form $0 \cdot \infty$ for $x = a$, we write the given function

$$f(x) \cdot \phi(x) = \frac{f(x)}{\frac{1}{\phi(x)}} \left(\text{or, } = \frac{\phi(x)}{\frac{1}{f(x)}}\right)$$

so as to cause it to take on one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, thus bringing it under § 123 or § 124.

**Ex. 1.** Evaluate $\sec 3x \cos 5x$ for $x = \frac{\pi}{2}$.

*Solution.* $\sec 3x \cos 5x\Big]_{x=\frac{\pi}{2}} = \infty \cdot 0$. $\therefore$ indeterminate.

Substituting $\frac{1}{\cos 3x}$ for $\sec 3x$, the function becomes $\frac{\cos 5x}{\cos 3x} = \frac{f(x)}{F(x)}$.

$$\frac{f\left(\frac{\pi}{2}\right)}{F\left(\frac{\pi}{2}\right)} = \frac{\cos 5x}{\cos 3x}\Bigg]_{x=\frac{\pi}{2}} = \frac{0}{0}. \quad \therefore \text{ indeterminate.}$$

$$\frac{f'\left(\frac{\pi}{2}\right)}{F'\left(\frac{\pi}{2}\right)} = \frac{-\sin 5x \cdot 5}{-\sin 3x \cdot 3}\Bigg]_{x=\frac{\pi}{2}} = -\frac{5}{3}. \quad \textit{Ans.}$$

**126. Evaluation of the indeterminate form $\infty - \infty$.** It is possible in general to transform the expression into a fraction which will assume either the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

**Ex. 1.** Evaluate $\sec x - \tan x$ for $x = \frac{\pi}{2}$.

*Solution.* $\sec x - \tan x\Big]_{x=\frac{\pi}{2}} = \infty - \infty$. $\therefore$ indeterminate.

By Trigonometry, $\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x} = \frac{f(x)}{F(x)}$.

$$\frac{f\left(\frac{\pi}{2}\right)}{F\left(\frac{\pi}{2}\right)} = \frac{1 - \sin x}{\cos x}\Bigg]_{x=\frac{\pi}{2}} = \frac{1-1}{0} = \frac{0}{0}. \quad \therefore \text{ indeterminate.}$$

$$\frac{f'\left(\frac{\pi}{2}\right)}{F'\left(\frac{\pi}{2}\right)} = \frac{-\cos x}{-\sin x}\Bigg]_{x=\frac{\pi}{2}} = \frac{0}{-1} = 0. \quad \textit{Ans.}$$

## EXAMPLES

Evaluate the following expressions by differentiation.*

1. $\underset{x=\infty}{\text{limit}} \frac{ax^2+b}{cx^2+d}$. *Ans.* $\frac{a}{c}$.

2. $\underset{x=0}{\text{limit}} \frac{\cot x}{\log x}$. $\infty$.

3. $\underset{x=\infty}{\text{limit}} \frac{\log x}{x^n}$. 0.

4. $\underset{\theta=\frac{\pi}{2}}{\text{limit}} \frac{\tan \theta}{\tan 3\theta}$. *Ans.* 3.

5. $\underset{\phi=\frac{\pi}{2}}{\text{limit}} \frac{\log\left(\phi - \frac{\pi}{2}\right)}{\tan \phi}$. 0.

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* In solving the remaining examples in this chapter it may be of assistance to the student to refer to § 37, pp. 35, 36, where many special forms *not indeterminate* are evaluated.

6. $\underset{y=\infty}{\text{limit}}\ \frac{y}{e^{ay}}$. *Ans.* 0.

7. $\underset{x=\frac{\pi}{2}}{\text{limit}}\ (\pi - 2x)\tan x$. 2.

8. $\underset{x=\infty}{\text{limit}}\ x \sin\frac{a}{x}$. $a$.

9. $\underset{x=0}{\text{limit}}\ x^n \log x$. [$n$ positive.] 0.

10. $\underset{x=\theta}{\text{limit}}\ (1 - \tan\theta)\sec 2\theta$. 1.

11. $\underset{\phi=a}{\text{limit}}\ (a^2 - \phi^2)\tan\frac{\pi\phi}{2a}$. $\frac{4a^2}{\pi}$.

12. $\underset{x=1}{\text{limit}}\left[\frac{2}{x^2-1} - \frac{1}{x-1}\right]$. $-\frac{1}{2}$.

13. $\underset{x=1}{\text{limit}}\left[\frac{1}{\log x} - \frac{x}{\log x}\right]$. $-1$.

14. $\underset{\theta=\frac{\pi}{2}}{\text{limit}}\ [\sec\theta - \tan\theta]$. 0.

15. $\underset{\phi=0}{\text{limit}}\left[\frac{2}{\sin^2\phi} - \frac{1}{1-\cos\phi}\right]$. $\frac{1}{2}$.

16. $\underset{y=1}{\text{limit}}\left[\frac{y}{y-1} - \frac{1}{\log y}\right]$. $\frac{1}{2}$.

17. $\underset{z=0}{\text{limit}}\left[\frac{\pi}{4z} - \frac{\pi}{2z(e^{\pi z}+1)}\right]$. $\frac{\pi^2}{8}$.

**127. Evaluation of the indeterminate forms $0^0$, $1^\infty$, $\infty^0$.** Given a function of the form

$$f(x)^{\phi(x)}.$$

In order that the function shall take on one of the above three forms, we must have for a certain value of $x$

$$f(x) = 0,\quad \phi(x) = 0,\ \text{giving } 0^0;$$

or, $$f(x) = 1,\quad \phi(x) = \infty,\ \text{giving } 1^\infty;$$

or, $$f(x) = \infty,\quad \phi(x) = 0,\ \text{giving } \infty^0.$$

Let $$y = f(x)^{\phi(x)};$$

taking the logarithm of both sides,

$$\log y = \phi(x)\log f(x).$$

In any of the above cases the logarithm of $y$ (the function) will take on the indeterminate form

$$0 \cdot \infty.$$

Evaluating this by the process illustrated in § 125 gives the limit of the logarithm of the function. This being equal to the logarithm of the limit of the function, the limit of the function is known.*

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* Thus, if limit $\log_e y = a$, then $y = e^a$.

**Ex. 1.** Evaluate $x^x$ when $x = 0$.

*Solution.* This function assumes the indeterminate form $0^0$ for $x = 0$.

Let $$y = x^x;$$

then $$\log y = x \log x = 0 \cdot -\infty, \quad \text{when } x = 0.$$

By § 125, p. 179, $$\log y = \frac{\log x}{\frac{1}{x}} = \frac{-\infty}{\infty}, \quad \text{when } x = 0.$$

By § 124, p. 177, $$\log y = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0, \quad \text{when } x = 0.$$

Since $y = x^x$, this gives $\log_e x^x = 0$; that is, $x^x = 1$. *Ans.*

**Ex. 2.** Evaluate $(1 + x)^{\frac{1}{x}}$ when $x = 0$.

*Solution.* This function assumes the indeterminate form $1^\infty$ for $x = 0$.

Let $$y = (1 + x)^{\frac{1}{x}};$$

then $$\log y = \frac{1}{x} \log (1 + x) = \infty \cdot 0, \quad \text{when } x = 0.$$

By § 125, p. 179, $$\log y = \frac{\log (1 + x)}{x} = \frac{0}{0}, \quad \text{when } x = 0.$$

By § 123, p. 174, $$\log y = \frac{\frac{1}{1+x}}{1} = \frac{1}{1+x} = 1, \quad \text{when } x = 0.$$

Since $y = (1 + x)^{\frac{1}{x}}$, this gives $\log_e (1 + x)^{\frac{1}{x}} = 1$; i.e. $(1 + x)^{\frac{1}{x}} = e$. *Ans.*

**Ex. 3.** Evaluate $(\cot x)^{\sin x}$ for $x = 0$.

*Solution.* This function assumes the indeterminate form $\infty^0$ for $x = 0$.

Let $$y = (\cot x)^{\sin x};$$

then $$\log y = \sin x \log \cot x = 0 \cdot \infty, \quad \text{when } x = 0.$$

By § 125, p. 179, $$\log y = \frac{\log \cot x}{\csc x} = \frac{\infty}{\infty}, \quad \text{when } x = 0.$$

By § 124, p. 177, $$\log y = \frac{\frac{-\csc^2 x}{\cot x}}{-\csc x \cot x} = \frac{\sin x}{\cos^2 x} = 0, \quad \text{when } x = 0.$$

Since $y = (\cot x)^{\sin x}$, this gives $\log_e (\cot x)^{\sin x} = 0$; i.e. $(\cot x)^{\sin x} = 1$. *Ans.*

## EXAMPLES

**Evaluate the following expressions by differentiation.**

1. $\underset{x=1}{\text{limit}}\, x^{\frac{1}{1-x}}$. *Ans.* $\frac{1}{e}$.

2. $\underset{x=0}{\text{limit}} \left(\frac{1}{x}\right)^{\tan x}$. $1$.

3. $\underset{\theta=\frac{\pi}{2}}{\text{limit}} (\sin\theta)^{\tan\theta}$. $1$.

4. $\underset{y=\infty}{\text{limit}} \left(1+\frac{a}{y}\right)^{y}$. $e^{a}$.

5. $\underset{x=0}{\text{limit}} (e^{x}+x)^{\frac{1}{x}}$. *Ans.* $e^{2}$.

6. $\underset{x=0}{\text{limit}} (\cot x)^{\frac{1}{\log x}}$. $\frac{1}{e}$.

7. $\underset{z=0}{\text{limit}} (1+nz)^{\frac{1}{z}}$. $e^{n}$.

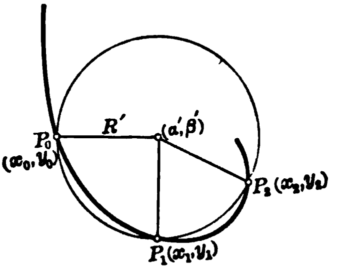
8. $\underset{\phi=1}{\text{limit}} \left(\tan\frac{\pi\phi}{4}\right)^{\tan\frac{\pi\phi}{2}}$. $\frac{1}{e}$.

9. $\underset{\theta=0}{\text{limit}} (\cos m\theta)^{\frac{n}{\theta^{2}}}$. $e^{-\frac{1}{2}nm^{2}}$.

# CHAPTER XVI

## CIRCLE OF CURVATURE. CENTER OF CURVATURE

**128. Circle of curvature.* Center of curvature.** If a circle be drawn through three points $P_0$, $P_1$, $P_2$ on a plane curve, and if $P_1$ and $P_2$ be made to approach $P_0$ along the curve as a limiting position, then the circle will in general approach in magnitude and position a limiting circle called the *circle of curvature of the curve at the point* $P_0$. The center of this circle is called the *center of curvature.*



Let the equation of the curve be

(1) $$y = f(x);$$

and let $x_0$, $x_1$, $x_2$ be the abscissas of the points $P_0$, $P_1$, $P_2$ respectively, $(\alpha', \beta')$ the coördinates of the center, and $R'$ the radius of the circle passing through the three points. Then the equation of the circle is

$$(x - \alpha')^2 + (y - \beta')^2 = R'^2;$$

and since the coördinates of the points $P_0$, $P_1$, $P_2$ must satisfy this equation, we have

(2) $$\begin{cases} (x_0 - \alpha')^2 + (y_0 - \beta')^2 - R'^2 = 0, \\ (x_1 - \alpha')^2 + (y_1 - \beta')^2 - R'^2 = 0, \\ (x_2 - \alpha')^2 + (y_2 - \beta')^2 - R'^2 = 0. \end{cases}$$

Now consider the *function of x* defined by

$$F(x) = (x - \alpha')^2 + (y - \beta')^2 - R'^2,$$

in which $y$ has been replaced by $f(x)$ from (1).

Then from equations (2) we get

$$F(x_0) = 0, \; F(x_1) = 0, \; F(x_2) = 0.$$

---

* Sometimes called the *osculating circle.*

Hence by Rolle's Theorem, p. 166, $F'(x)$ must vanish for at least two values of $x$, one lying between $x_0$ and $x_1$, say $x'$, and the other lying between $x_1$ and $x_2$, say $x''$; that is,

$$F'(x')=0,\quad F'(x'')=0.$$

Again, for the same reason, $F''(x)$ must vanish for some value of $x$ between $x'$ and $x''$, say $x_3$; hence

$$F''(x_3)=0.$$

Therefore the elements $\alpha'$, $\beta'$, $R'$ of the circle passing through the points $P_0$, $P_1$, $P_2$ must satisfy the three equations

$$F(x_0)=0,\quad F'(x')=0,\quad F''(x_3)=0.$$

Now let the points $P_1$ and $P_2$ approach $P_0$ as a limiting position; then $x_1$, $x_2$, $x'$, $x''$, $x_3$ will all approach $x_0$ as a limit, and the elements $\alpha$, $\beta$, $R$ of the osculating circle are therefore determined by the three equations

$$F(x_0)=0,\quad F'(x_0)=0,\quad F''(x_0)=0;$$

or, dropping the subscripts, what is the same thing,

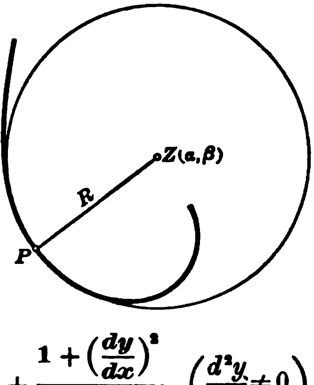
$$(A)\qquad (x-\alpha)^2+(y-\beta)^2=R^2,$$

$$(B)\qquad (x-\alpha)+(y-\beta)\frac{dy}{dx}=0,\text{ differentiating } (A).$$

$$(C)\qquad 1+\left(\frac{dy}{dx}\right)^2+(y-\beta)\frac{d^2y}{dx^2}=0,\text{ differentiating } (B).$$

Solving $(B)$ and $(C)$ for $x-\alpha$ and $y-\beta$, we get, $\left(\frac{d^2y}{dx^2}\neq 0\right)$,

$$(D)\quad \begin{cases} x-\alpha=\dfrac{\dfrac{dy}{dx}\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}}, \\[2ex] y-\beta=-\dfrac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}; \end{cases}$$



hence the *coördinates of the center of curvature are*

$$(E)\qquad \alpha=x-\frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}};\quad \beta=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.\quad \left(\frac{d^2y}{dx^2}\neq 0\right)$$

Substituting the values of $x-\alpha$ and $y-\beta$ from $(D)$ in $(A)$, and solving for $R$, we get

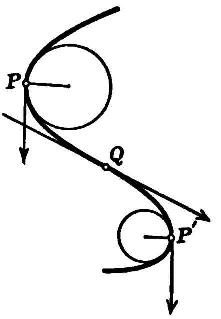
$$R=\pm\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

which is identical with (40), p. 163. Hence

**Theorem.** *The radius of the circle of curvature equals the radius of curvature.*

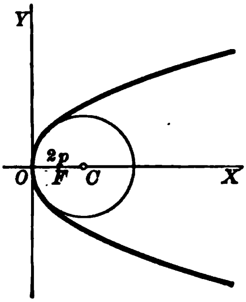
From (23), p. 136, we know that at a point of inflection (as $Q$ in the figure)

$$\frac{d^2y}{dx^2}=0.$$



Therefore, by (38), p. 161, the curvature $K=0$; and from (40), p. 163, and $(E)$, p. 184, we see that in general $\alpha$, $\beta$, $R$ increase without limit as the second derivative approaches zero. That is, if we suppose $P$ with its tangent to move along the curve to $P'$, at the point of inflection $Q$ the curvature is zero, the rotation of the tangent is momentarily arrested, and as the direction of rotation changes, the center of curvature moves out indefinitely and the radius of curvature becomes infinite.

Ex. 1. Find the coördinates of the center of curvature of the parabola $y^2=4px$ corresponding (a) to any point on the curve; (b) to the vertex.



*Solution.* $\frac{dy}{dx}=\frac{2p}{y};\ \frac{d^2y}{dx^2}=-\frac{4p^2}{y^2}.$

(a) Substituting in $(E)$,

$$\alpha=x+\frac{y^2+4p^2}{y^2}\cdot\frac{2p}{y}\cdot\frac{y^3}{4p^2}=3x+2p.$$

$$\beta=y-\frac{y^2+4p^2}{y^2}\cdot\frac{y^3}{4p^2}=-\frac{y^3}{4p^2}.$$

Therefore $\left(3x+2p,\ -\frac{y^3}{4p^2}\right)$ is the center of curvature corresponding to any point on the curve.

(b) $(2p, 0)$ is the center of curvature corresponding to the vertex $(0, 0)$.

**129. Center of curvature the limiting position of the intersection of normals at neighboring points.**

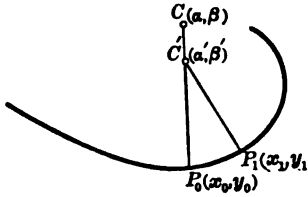
Let the equation of a curve be

$$(A) \qquad y = f(x).$$

The equations of the normals to the curve at two neighboring points $P_0$ and $P_1$ are*

$$(x_0 - X) + (y_0 - Y)\frac{dy_0}{dx_0} = 0,$$

$$(x_1 - X) + (y_1 - Y)\frac{dy_1}{dx_1} = 0.$$



If the normals intersect at $C'(\alpha', \beta')$, the coördinates of this point must satisfy both equations, giving

$$(B) \qquad \begin{cases} (x_0 - \alpha') + (y_0 - \beta')\dfrac{dy_0}{dx_0} = 0, \\ (x_1 - \alpha') + (y_1 - \beta')\dfrac{dy_1}{dx_1} = 0. \end{cases}$$

Now consider the *function of* $x$ defined by

$$\phi(x) = (x - \alpha') + (y - \beta')\frac{dy}{dx},$$

in which $y$ has been replaced by $f(x)$ from $(A)$.

Then equations $(B)$ show that

$$\phi(x_0) = 0, \quad \phi(x_1) = 0.$$

But then by Rolle's Theorem, p. 166, $\phi'(x)$ must vanish for some value of $x$ between $x$ and $x_1$, say $x'$. Therefore $\alpha'$ and $\beta'$ are determined by the two equations

$$\phi(x_0) = 0, \quad \phi'(x') = 0.$$

If now $P_1$ approaches $P_0$ as a limiting position, then $x'$ approaches $x_0$, giving

$$\phi(x_0) = 0, \quad \phi'(x_0) = 0;$$

and $C''$ $(\alpha', \beta')$ will approach as a limiting position the center of curvature $C$ $(\alpha, \beta)$ corresponding to $P_0$ on the curve. For, if we drop the subscripts and write the last two equations in the form

---

* From (2), p. 90, $X$ and $Y$ being the variable coördinates.

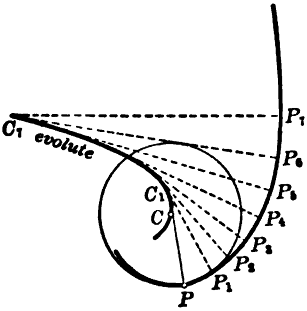
$$(x-\alpha')+(y-\beta')\frac{dy}{dx}=0,$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-\beta')\frac{d^2y}{dx^2}=0,$$

it is evident that solving for $\alpha'$ and $\beta'$ will give the same results as solving $(B)$ and $(C)$, p. 184, for $\alpha$ and $\beta$. Hence

**Theorem.** *The center of curvature $C$ corresponding to a point $P$ on a curve is the limiting position of the intersection of the normal to the curve at $P$ with a neighboring normal.*

**130. Evolutes.** The locus of the centers of curvature of a given curve is called the *evolute* of that curve. Consider the circle of curvature corresponding to a point $P$ on a curve. If $P$ moves along the given curve, we may suppose the corresponding circle of curvature to roll along the curve with it, its radius varying so as to be always equal to the radius of curvature of the curve at the point $P$. The curve $CC_7$ described by the center of the circle is the evolute of $PP_7$.



Formula $(E)$, p. 184, gives the coördinates of any point $(\alpha, \beta)$ on the evolute expressed in terms of the coördinates of the corresponding point $(x, y)$ of the given curve. But $y$ is a function of $x$, therefore

$$\alpha=x-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}, \quad \beta=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

give us at once *the parametric equations of the evolute in terms of the parameter $x$.*

To find the ordinary rectangular equation of the evolute we eliminate $x$ between the two expressions. No general process of elimination can be given that will apply in all cases, the method to be adopted depending on the form of the given equation. In a large number of cases, however, the student can find the rectangular equation of the evolute by taking the following steps.

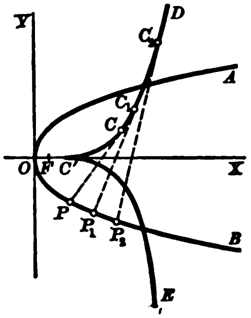
**General directions for finding the evolute.**

**First step.** *Find $\alpha$ and $\beta$ from* $(E)$, *p.* 184.

**Second step.** *Solve the two resulting equations for $x$ and $y$ in terms of $\alpha$ and $\beta$.*

**Third step.** *Substitute these values of $x$ and $y$ in the given equation. This gives a relation between the variables $\alpha$ and $\beta$ which is the equation of the evolute.*

Ex. 1. Find the equation of the evolute of the parabola of $y^2 = 4px$.



*Solution.* $\frac{dy}{dx} = \frac{2p}{y}, \quad \frac{d^2y}{dx^2} = -\frac{4p^2}{y^3}.$

*First step.* $\alpha = 3x + 2p, \quad \beta = -\frac{y^3}{4p^2}.$

*Second step.* $x = \frac{\alpha - 2p}{3}, \quad y = -(4p^2\beta)^{\frac{1}{3}}.$

*Third step.* $(4p^2\beta)^{\frac{2}{3}} = 4p\left(\frac{\alpha - 2p}{3}\right)$, or,

$$p\beta^2 = \frac{4}{27}(\alpha - 2p)^3.$$

Remembering that $\alpha$ denotes the abscissa and $\beta$ the ordinate of a rectangular system of coördinates, we see that the evolute of the parabola $AOB$ is the semicubical parabola $DC'E$; the centers of curvature for $O$, $P$, $P_1$, $P_2$ being at $C'$, $C$, $C_1$, $C_2$ respectively.

Ex. 2. Find the equation of the evolute of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

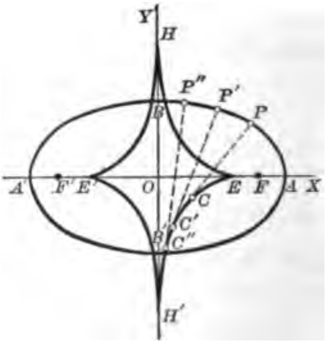
*Solution.* $\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$

*First step.* $\alpha = \frac{(a^2 - b^2)x^3}{a^4},$

$\beta = -\frac{(a^2 - b^2)y^3}{b^4}.$

*Second step.* $x = \left(\frac{a^4\alpha}{a^2 - b^2}\right)^{\frac{1}{3}},$

$y = -\left(\frac{b^4\beta}{a^2 - b^2}\right)^{\frac{1}{3}}.$



*Third step.* $(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$

the equation of the evolute $EHE'H'$ of the ellipse $ABA'B'$. $E$, $E'$, $H'$, $H$ are the centers of curvature corresponding to the points $A$, $A'$, $B$, $B'$ on the curve, and $C$, $C'$, $C''$ correspond to the points $P$, $P'$, $P''$.

When the equations of the curve are given in parametric form, we proceed to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, as on pp. 154, 155, from

$$(A) \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

$$(B) \qquad \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3};$$

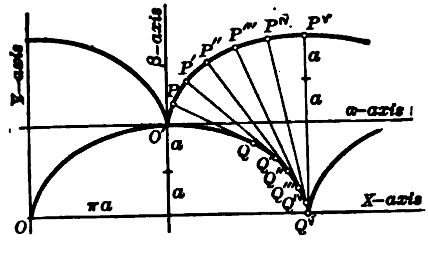
and then substitute the results in formulas (*E*), p. 184. This gives the parametric equations of the evolute in terms of the same parameter that occurs in the given equations.

**Ex. 3.** Find the parametric equations of the evolute of the cycloid.

$$(C) \qquad \begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$$

*Solution.* As in Ex. 2, p. 163, we get

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}, \quad \frac{d^2y}{dx^2} = -\frac{1}{(1 - \cos t)^2}.$$



Substituting these results in formulas (*E*), p. 184, we get

$$(D) \qquad \begin{cases} \alpha = a(t + \sin t), \\ \beta = -a(1 - \cos t). \end{cases} \quad \textit{Ans.}$$

NOTE. If we eliminate $t$ between equations (*D*) there results the rectangular equation of the evolute $OO'Q^v$ referred to the axes $O'\alpha$ and $O'\beta$. The coördinates of $O$ with respect to these axes are $(-\pi a, -2a)$. Let us transform equations (*D*) to the new set of axes $OX$ and $OY$. Then

$$\alpha = x - \pi a, \ \beta = y - 2a, \ t = t' - \pi.$$

Substituting in (*D*) and reducing, the equations of the evolute become

$$(E) \qquad \begin{cases} x = a(t' - \sin t'), \\ y = a(1 - \cos t'). \end{cases}$$

Since (*E*) and (*C*) are identical in form, we have:

*The evolute of a cycloid is itself a cycloid whose generating circle equals that of the given cycloid.*

**131. Properties of the evolute.** Differentiating $\alpha$ and $\beta$ from $(E)$, p. 184, with respect to $x$ gives

$$(A)\qquad \frac{d\alpha}{dx} = -\frac{dy}{dx}\cdot\frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}}{\left(\frac{d^2y}{dx^2}\right)^2},$$

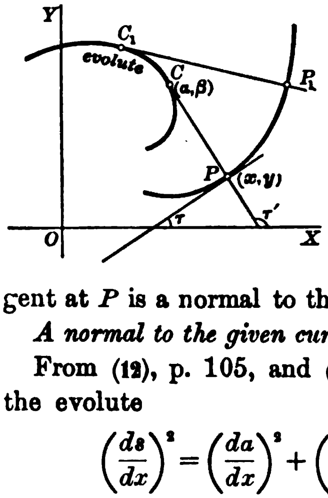
$$(B)\qquad \frac{d\beta}{dx} = \frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}}{\left(\frac{d^2y}{dx^2}\right)^2}.$$

Dividing $(B)$ by $(A)$, member for member,

$$(C)\qquad \frac{d\beta}{d\alpha} = -\frac{1}{\frac{dy}{dx}}.$$

But $\frac{d\beta}{d\alpha} = \tan\tau' =$ slope of tangent to the evolute at $C$, and

$\frac{dy}{dx} = \tan\tau =$ slope of tangent to the given curve at the corresponding point $P$.



Substituting the last two results in $(C)$, we get

$$\tan\tau' = -\frac{1}{\tan\tau}.$$

Since the slope of one tangent is the negative reciprocal of the slope of the other, they are perpendicular. But a line perpendicular to the tangent at $P$ is a normal to the curve. Hence

*A normal to the given curve is a tangent to its evolute.*

From (12), p. 105, and $(A)$ and $(B)$, we have for an arc $s$ of the evolute

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{d\alpha}{dx}\right)^2 + \left(\frac{d\beta}{dx}\right)^2$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right]\left(\frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}}{\left(\frac{d^2y}{dx^2}\right)^2}\right)^2.$$

But this is identically the result we get by differentiating $R$, (40), p. 163, with respect to $x$ and then squaring. Therefore

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dR}{dx}\right)^2,$$

or,

$$ds = \pm dR.$$

That is, *the radius of curvature of the given curve increases or decreases as fast as the arc of the evolute increases.* In our figure this means that

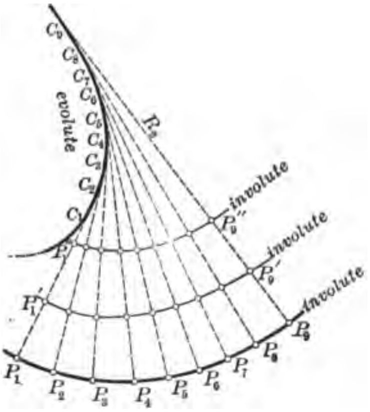
$$P_1C_1 - PC = \text{arc } CC_1.$$

*The length of an arc of the evolute is equal to the difference between the radii of curvature of the given curve which are tangent to this arc at its extremities.**

Thus in Ex. 3, p. 189, we observe that if we fold $Q^{v}P^{v}$ ($= 4a$) over to the left on the evolute, $P^{v}$ will reach to $O'$, and we have:

*The length of one arc of the cycloid (as $OO'Q^{v}$) is eight times the length of the radius of the generating circle.*

**132. Involutes and their mechanical construction.** Let a flexible ruler be bent in the form of the curve $C_1C_9$, the evolute of the curve $P_1P_9$, and suppose a string of length $R_9$ with one end fastened at $C_9$ to be wrapped around the ruler (or curve). It is clear from the results of the last section that when the string is unwound and kept taut the free end will describe the curve $P_1P_9$. Hence the name *evolute.*



The curve $P_1P_9$ is said to be an *involute* of $C_1C_9$. Obviously any point on the string will describe an involute, so that a given curve has an infinite number of involutes but only one evolute.

The involutes $P_1P_9$, $P_1'P_9'$, $P_1''P_9''$ are called *parallel curves* since the distance between any two of them measured along their common normals is constant.

The student should observe how the parabola and ellipse on p. 188 may be constructed in this way from their evolutes.

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* It is assumed that $\frac{dR}{dx}$ does not change sign.

## EXAMPLES

Find the coördinates of the center of curvature and the equation of the evolute of each of the following curves.

1. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. *Ans.* $\alpha = \frac{(a^2 + b^2)x^3}{a^4}$, $\beta = -\frac{(a^2 + b^2)y^3}{b^4}$; evolute $(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$.

2. The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\alpha = x + 3x^{\frac{1}{3}}y^{\frac{2}{3}}$, $\beta = y + 3x^{\frac{2}{3}}y^{\frac{1}{3}}$; evolute $(\alpha + \beta)^{\frac{2}{3}} + (\alpha - \beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

3. The cycloid $x = r \operatorname{arcvers} \frac{y}{r} - \sqrt{2ry - y^2}$. *Ans.* $\alpha = x + 2\sqrt{2ry - y^2}$, $\beta = -y$; evolute $\alpha = r \operatorname{arcvers}\left(-\frac{\beta}{r}\right) + \sqrt{-2r\beta - \beta^2}$.

4. The semicubical parabola $x^3 = ay^2$. *Ans.* $\alpha = -x - \frac{9x^2}{2a}$, $\beta = 4\left(x + \frac{a}{3}\right)\sqrt{\frac{x}{a}}$; evolute $729a\alpha^2 = 16\left[2a + \sqrt{a^2 - 18a\alpha}\right]^2\left[\sqrt{a^2 - 18a\alpha} - a\right]$.

5. The tractrix $x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$. *Ans.* $\alpha = a \log \frac{a + \sqrt{a^2 - y^2}}{y}$, $\beta = \frac{a^2}{y}$; evolute $\beta = \frac{a}{2}\left(e^{\frac{\alpha}{a}} + e^{-\frac{\alpha}{a}}\right)$.

6. The catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. *Ans.* $\alpha = x - \frac{y}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$, $\beta = 2y$; evolute $\alpha = a \log \frac{\beta \pm (\beta^2 - 4a^2)^{\frac{1}{2}}}{2a} \mp \frac{\beta}{4a}(\beta^2 - 4a^2)^{\frac{1}{2}}$.

7. Find the coördinates of the center of curvature of the cubical parabola $y^3 = a^2x$. *Ans.* $\alpha = \frac{a^4 + 15y^4}{6a^2y}$, $\beta = \frac{a^4y - 9y^5}{2a^4}$.

8. Show that in the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ we have the relation $\alpha + \beta = 3(x + y)$.

9. Find the equation of the evolute of the cissoid $y^2 = \frac{x^3}{2a - x}$. *Ans.* $4096a^3\alpha + 1152a^2\beta^2 + 27\beta^4 = 0$.

10. Given the equation of the equilateral hyperbola $2xy = a^2$; show that

$$\alpha + \beta = \frac{(y + x)^3}{a^2}, \quad \alpha - \beta = \frac{(y - x)^3}{a}.$$

From this derive the equation of the evolute $(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

Find the parametric equations of the evolutes of the following curves in terms of the parameter $t$.

11. The hypocycloid $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$ *Ans.* $\begin{cases} \alpha = a\cos^3 t + 3a\cos t\sin^2 t, \\ \beta = 3a\cos^2 t\sin t + a\sin^3 t. \end{cases}$

12. The curve $\begin{cases} x = 3t^2, \\ y = 3t - t^3. \end{cases}$ *Ans.* $\begin{cases} \alpha = \frac{3}{2}(1 + 2t^2 - t^4), \\ \beta = -4t^3. \end{cases}$

13. The curve $\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t). \end{cases}$ *Ans.* $\begin{cases} \alpha = a\cos t, \\ \beta = a\sin t. \end{cases}$

# CHAPTER XVII

## PARTIAL DIFFERENTIATION

**133. Continuous functions of two or more independent variables.** A function $f(x, y)$ of two independent variables $x$ and $y$ is defined as continuous for the values $(a, b)$ of $(x, y)$ when

$$\underset{\substack{x=a\\y=b}}{\text{limit}} f(x, y) = f(a, b),$$

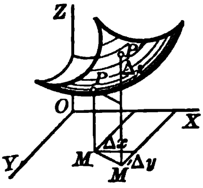
no matter in what way $x$ and $y$ approach their respective limits $a$ and $b$. This definition is sometimes roughly summed up in the statement that *a very small change in one or both of the independent variables shall produce a very small change in the value of the function.**

We may illustrate this geometrically by considering the surface represented by the equation

$$z = f(x, y).$$

Consider a fixed point $P$ on the surface where $x = a$ and $y = b$. Denote by $\Delta x$ and $\Delta y$ the increments of the independent variables $x$ and $y$, and by $\Delta z$ the corresponding increment of the dependent variable $z$, the coördinates of $P'$ being

$$(x + \Delta x,\ y + \Delta y,\ z + \Delta z).$$



At $P$ the value of the function is

$$z = f(a, b) = MP.$$

If the function is continuous at $P$, then however $\Delta x$ and $\Delta y$ may approach the limit zero, $\Delta z$ will also approach the limit zero. That is, $M'P'$ will approach coincidence with $MP$, the point $P'$ approaching the point $P$ on the surface from any direction whatever.

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* This will be better understood if the student again reads over § 33, p. 22, on continuous functions of a single variable.

A similar definition holds for a continuous function of more than two independent variables.

In what follows, only values of the independent variables are considered for which a function is continuous.

**134. Partial derivatives.** Since $x$ and $y$ are independent in

$$z = f(x, y),$$

$x$ may be supposed to vary while $y$ remains constant, or the reverse.

The derivative of $z$ with respect to $x$ when $x$ varies and $y$ remains constant* is called the *partial derivative of $z$ with respect to $x$*, and is denoted by the symbol $\frac{\partial z}{\partial x}$. We may then write

$$(A) \qquad \frac{\partial z}{\partial x} = \underset{\Delta x = 0}{\text{limit}} \left[\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}\right].$$

Similarly, when $x$ remains constant* and $y$ varies, *the partial derivative of $z$ with respect to $y$* is

$$(B) \qquad \frac{\partial z}{\partial y} = \underset{\Delta y = 0}{\text{limit}} \left[\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\right].$$

$\frac{\partial z}{\partial x}$ is also written $\frac{\partial}{\partial x} f(x, y)$ or $\frac{\partial f}{\partial x}$; similarly

$\frac{\partial z}{\partial y}$ is also written $\frac{\partial}{\partial y} f(x, y)$ or $\frac{\partial f}{\partial y}$.

In order to avoid confusion the round $\partial$† has been generally adopted to indicate partial differentiation. Other notations, however, which are in use are

$$\left(\frac{dz}{dx}\right), \left(\frac{dz}{dy}\right); \; f_x'(x, y), f_y'(x, y); \; f_x(x, y), f_y(x, y); \; D_x f, D_y f; \; z_x, z_y.$$

Our notation may be extended to a function of any number of independent variables. Thus, if

$$u = F(x, y, z),$$

then we have the three partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}; \text{ or } \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

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* The constant values are substituted in the function before differentiating.

† Introduced by Jacobi (1804–1851).

**Ex. 1.** Find the partial derivatives of $z = ax^2 + 2bxy + cy^2$.

*Solution.* $\dfrac{\partial z}{\partial x} = 2ax + 2by$, treating $y$ as a constant,

$\dfrac{\partial z}{\partial y} = 2bx + 2cy$, treating $x$ as a constant.

**Ex. 2.** Find the partial derivatives of $u = \sin(ax + by + cz)$.

*Solution.* $\dfrac{\partial u}{\partial x} = a\cos(ax + by + cz)$, treating $y$ and $z$ as constants,

$\dfrac{\partial u}{\partial y} = b\cos(ax + by + cz)$, treating $x$ and $z$ as constants,

$\dfrac{\partial u}{\partial z} = c\cos(ax + by + cz)$, treating $y$ and $x$ as constants.

Again turning to the function

$$z = f(x, y),$$

we have by $(A)$ defined $\dfrac{\partial z}{\partial x}$ as the limit of the ratio of the increment of the function ($y$ being constant) to the increment of $x$, as the increment of $x$ approaches the limit zero. Similarly $(B)$ has defined $\dfrac{\partial z}{\partial y}$.

It is evident, however, that if we look upon these partial derivatives from the point of view of § 106, p. 148, then

$$\frac{\partial z}{\partial x}$$

may be considered as the ratio of the time rates of change of $z$ and $x$ when $y$ is constant, and

$$\frac{\partial z}{\partial y}$$

as the ratio of the time rates of change of $z$ and $y$ when $x$ is constant.

**135. Partial derivatives interpreted geometrically.** Let the equation of the surface shown in the figure (next page) be

$$z = f(x, y).$$

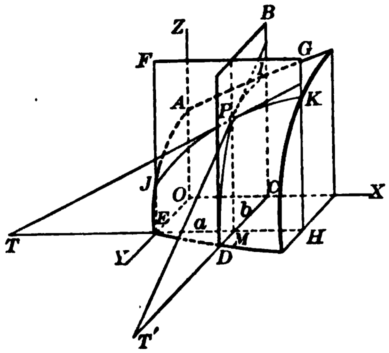
Pass a plane $EFGH$ through the point $P$ (where $x = a$ and $y = b$) on the surface parallel to the $XOZ$ plane. Since the equation of this plane is

$$y = b,$$

the equation of the section $JPK$ cut out of the surface is

$$z = f(x, b),$$

if we consider $EF$ as the axis of $Z$ and $EH$ as the axis of $X$. In this plane $\frac{\partial z}{\partial x}$ means the same as $\frac{dz}{dx}$, and we have



$$\frac{\partial z}{\partial x} = \tan MTP$$

$$= \textit{slope of section } JK \textit{ at } P.$$

Similarly, if we pass the plane $BCD$ through $P$ parallel to the $YOZ$ plane, its equation is

$$x = a,$$

and for the section $DPI$, $\frac{\partial z}{\partial y}$ means the same as $\frac{dz}{dy}$. Hence

$$\frac{\partial z}{\partial y} = \tan MT'P = \textit{slope of section } DI \textit{ at } P.$$

Ex. 1. Given the ellipsoid $\frac{x^2}{24} + \frac{y^2}{12} + \frac{z^2}{6} = 1$; find the slope of the section of the ellipsoid made (a) by the plane $y = 1$ at the point where $x = 4$ and $z$ is positive; (b) by the plane $x = 2$ at the point where $y = 3$ and $z$ is positive.

*Solution.* Considering $y$ as constant,

$$\frac{2x}{24} + \frac{2z}{6}\frac{\partial z}{\partial x} = 0, \text{ or, } \frac{\partial z}{\partial x} = -\frac{x}{4z}.$$

When $x$ is constant, $\frac{2y}{12} + \frac{2z}{6}\frac{\partial z}{\partial y} = 0$, or, $\frac{\partial z}{\partial y} = -\frac{y}{2z}$.

(a) When $y = 1$ and $x = 4$, $z = \sqrt{\frac{3}{2}}$. $\therefore \frac{\partial z}{\partial x} = -\sqrt{\frac{2}{3}}$. *Ans.*

(b) When $x = 2$ and $y = 3$, $z = \frac{1}{\sqrt{2}}$. $\therefore \frac{\partial z}{\partial y} = -\frac{3}{2}\sqrt{2}$. *Ans.*

## EXAMPLES

1. $u = x^3 + 3x^2y - y^3$. *Ans.* $\frac{\partial u}{\partial x} = 3x^2 + 6xy$;

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2.$$

2. $u = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$. *Ans.* $\frac{\partial u}{\partial x} = 2Ax + By + D$;

$$\frac{\partial u}{\partial y} = Bx + 2Cy + E.$$

3. $u = (ax^2 + by^2 + cz^2)^n$. *Ans.* $\frac{\partial u}{\partial x} = \frac{2anxu}{ax^2 + by^2 + cz^2}$; $\frac{\partial u}{\partial y} = \frac{2bnyu}{ax^2 + by^2 + cz^2}$.

4. $u = \arcsin \frac{x}{y}$. *Ans.* $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}}$; $\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2 - x^2}}$.

5. $u = x^y$. *Ans.* $\frac{\partial u}{\partial x} = yx^{y-1}$; $\frac{\partial u}{\partial y} = x^y \log x$.

6. $u = ax^3y^2z + bxy^3z^4 + cy^6 + dxz^3$. *Ans.* $\frac{\partial u}{\partial x} = 3ax^2y^2z + by^3z^4 + dz^3$; $\frac{\partial u}{\partial y} = 2ax^3yz + 3bxy^2z^4 + 6cy^5$; $\frac{\partial u}{\partial z} = ax^3y^2 + 4bxy^3z^3 + 3dxz^2$.

7. $u = x^3y^2 - 2xy^4 + 3x^2y^3$; show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 5u$.

8. $u = \frac{xy}{x + y}$; show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

9. $u = (y - z)(z - x)(x - y)$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

10. $u = \log(e^x + e^y)$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$.

11. $u = \frac{e^{xy}}{e^x + e^y}$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (x + y - 1)u$.

12. $u = x^y y^x$; show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (x + y + \log u)u$.

13. $u = \log(x^3 + y^3 + z^3 - 3xyz)$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$.

14. $u = e^x \sin y + e^y \sin x$; show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{2x} + e^{2y} + 2e^{x+y}\sin(x + y).$$

15. $u = \log(\tan x + \tan y + \tan z)$; show that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$

16. Let $y$ be the altitude of a right circular cone and $x$ the radius of its base. Show (a) that if the base remains constant, the volume changes $\frac{1}{3}\pi x^2$ times as fast as the altitude; (b) that if the altitude remains constant, the volume changes $\frac{2}{3}\pi xy$ times as fast as the radius of the base.

17. A point moves on the elliptic paraboloid $z = \frac{x^2}{9} + \frac{y^2}{4}$ and also in a plane parallel to the $XOZ$ plane. When $x = 3$ ft. and is increasing at the rate of 9 ft. per second, find (a) the time rate of change of $z$; (b) the magnitude of the velocity of the point; (c) the direction of its motion.

*Ans.* (a) $v_z = 6$ ft. per sec.; (b) $v = 3\sqrt{5}$ ft. per sec.;
(c) $\tau = \text{arc tan } \frac{2}{3}$, the angle made with the $XOY$ plane.

18. If, on the surface of Ex. 17, the point moves in a plane parallel to the plane $YOZ$, find, when $y = 2$ and increases at the rate of 5 ft. per sec., (a) the time rate of change of $z$; (b) the magnitude of the velocity of the point; (c) the direction of its motion.

*Ans.* (a) 5 ft. per sec.; (b) $5\sqrt{2}$ ft. per sec.;
(c) $\tau = \frac{\pi}{4}$, the angle made with the plane $XOY$.

**136. Total derivatives.** We have already, on page 57, considered the differentiation of a function of one function of a single independent variable. Thus, if

$$y = f(v) \text{ and } v = \phi(x),$$

it was shown that

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

We shall next consider a function of two variables, both of which depend on a single independent variable. Consider the function

$$u = f(x, y),$$

where $x$ and $y$ are functions of a third variable $t$.

Let $t$ take on the increment $\Delta t$, and let $\Delta x$, $\Delta y$, $\Delta u$ be the corresponding increments of $x$, $y$, $u$ respectively. Then the quantity

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$$

is called the *total increment* of $u$.

Adding and subtracting $f(x, y + \Delta y)$ in the second member,

$$(A)\quad \Delta u = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)].$$

Applying the Theorem of Mean Value, (44), p. 168, to each of the two differences on the right-hand side of $(A)$, we get, for the first difference,

$$(B)\quad f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = f_x'(x + \theta_1 \cdot \Delta x, y + \Delta y)\,\Delta x.$$

[$a = x$, $\Delta a = \Delta x$, and since $x$ varies while $y + \Delta y$ remains constant, we get the partial derivative with respect to $x$.]

For the second difference we get

$$(C) \qquad f(x,\ y+\Delta y)-f(x,\ y)=f_y'(x,\ y+\theta_2\cdot\Delta y)\Delta y.$$

$\left[a=y,\ \Delta a=\Delta y, \text{ and since } y \text{ varies while } x \text{ remains constant, we get the partial derivative with respect to } y.\right]$

Substituting (*B*) and (*C*) in (*A*) gives

$$(D) \quad \Delta u=f_x'(x+\theta_1\cdot\Delta x,\ y+\Delta y)\Delta x+f_y'(x,\ y+\theta_2\cdot\Delta y)\Delta y,$$

where $\theta_1$ and $\theta_2$ are positive proper fractions. Dividing (*D*) by $\Delta t$,

$$(E) \quad \frac{\Delta u}{\Delta t}=f_x'(x+\theta_1\cdot\Delta x,\ y+\Delta y)\frac{\Delta x}{\Delta t}+f_y'(x,\ y+\theta_2\cdot\Delta y)\frac{\Delta y}{\Delta t}.$$

Now let $\Delta t$ approach zero as a limit, then

$$(F) \qquad \frac{du}{dt}=f_x'(x,\ y)\frac{dx}{dt}+f_y'(x,\ y)\frac{dy}{dt}.$$

$\left[\begin{array}{c}\text{Since } \Delta x \text{ and } \Delta y \text{ converge to zero with } \Delta t, \text{ we get} \\ \underset{\Delta t=0}{\text{limit}} f_x'(x+\theta_1\cdot\Delta x,\ y+\Delta y)=f_x'(x,y), \text{ and } \underset{\Delta t=0}{\text{limit}} f_y'(x,\ y+\theta_2\cdot\Delta y)=f_y'(x,\ y), \\ f_x'(x,\ y) \text{ and } f_y'(x,\ y) \text{ being assumed continuous.}\end{array}\right]$

Replacing $f(x,\ y)$ by $u$ in (*F*), we get the *total derivative*

$$(49) \qquad \frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}.$$

In the same way, if

$$u=f(x,\ y,\ z),$$

and $x$, $y$, $z$ are all functions of $t$, we get

$$(50) \qquad \frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt},$$

and so on for any number of variables.*

In (49) we may suppose $t=x$; then $y$ is a function of $x$, and $u$ is really a function of the one variable $x$, giving

$$(51) \qquad \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}.$$

In the same way from (50) we have

$$(52) \qquad \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial u}{\partial z}\cdot\frac{dz}{dx}.$$

---

* This is really only a special case of a general theorem which may be stated as follows: If $u$ is a function of the independent variables $x$, $y$, $z$, $\cdots$, each of these in turn being a function of the independent variables $r$, $s$, $t$, $\cdots$, then (with certain assumptions as to continuity)

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}+\cdots,$$

and similar expressions hold for $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$, etc.

The student should observe that $\frac{\partial u}{\partial x}$ and $\frac{du}{dx}$ have quite different meanings. The partial derivative $\frac{\partial u}{\partial x}$ is formed on the supposition that the *particular variable x alone varies*, while

$$\frac{du}{dx} = \underset{\Delta x = 0}{\text{limit}} \left(\frac{\Delta u}{\Delta x}\right),$$

where $\Delta u$ is the *total increment of u* caused by changes in *all the variables*, these increments being due to the change $\Delta x$ in the independent variable. In contradistinction to partial derivatives, $\frac{du}{dt}, \frac{du}{dx}$ are called *total derivatives* with respect to $t$ and $x$ respectively.*

Ex. 1. Given $u = \sin\frac{x}{y}$, $x = e^t$, $y = t^2$; find $\frac{du}{dt}$.

*Solution.* $\frac{\partial u}{\partial x} = \frac{1}{y}\cos\frac{x}{y}$, $\frac{\partial u}{\partial y} = -\frac{x}{y^2}\cos\frac{x}{y}$; $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = 2t$.

Substituting in (49), $\frac{du}{dt} = (t-2)\frac{e^t}{t^3}\cos\frac{e^t}{t^2}$. *Ans.*

Ex. 2. Given $u = e^{ax}(y - z)$, $y = a\sin x$, $z = \cos x$; find $\frac{du}{dx}$.

*Solution.* $\frac{\partial u}{\partial x} = ae^{ax}(y - z)$, $\frac{\partial u}{\partial y} = e^{ax}$, $\frac{\partial u}{\partial z} = -e^{ax}$; $\frac{dy}{dx} = a\cos x$, $\frac{dz}{dx} = -\sin x$.

Substituting in (52),

$$\frac{du}{dx} = ae^{ax}(y - z) + ae^{ax}\cos x + e^{ax}\sin x = e^{ax}(a^2 + 1)\sin x. \quad \textit{Ans.}$$

NOTE. In examples like the above, $u$ could, by substitution, be found explicitly in terms of the independent variable and then differentiated directly, but generally this process would be longer and in many cases could not be used at all.

**137. Total differentials.** Multiplying (49) and (50) through by $dt$, we get, (§ 104, p. 144),

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \tag{53}$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz; \tag{54}$$

---

* It should be observed that $\frac{\partial u}{\partial x}$ has a perfectly definite value for any point $(x, y)$, while $\frac{du}{dx}$ depends not only on the point $(x, y)$ but also on the particular direction chosen to reach that point. Hence

$\frac{\partial u}{\partial x}$ is called a point function; while

$\frac{du}{dx}$ is not called a point function unless it is agreed to approach the point from some particular direction.

and so on.[*] Equations (53) and (54) define the quantity $du$, which is called a *total differential of u* or a *complete differential*,

and
$$\frac{\partial u}{\partial x}dx,\ \frac{\partial u}{\partial y}dy,\ \frac{\partial u}{\partial z}dz$$

are called *partial differentials*. These partial differentials are sometimes denoted by $d_xu$, $d_yu$, $d_zu$, so that (54) is also written

$$du = d_xu + d_yu + d_zu.$$

Ex. 1. Given $u = \text{arc}\tan\frac{y}{x}$; find $du$.

*Solution.*
$$\frac{\partial u}{\partial x} = -\frac{y}{x^2+y^2},\ \frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}.$$

Substituting in (53),
$$du = \frac{xdy - ydx}{x^2+y^2}.\quad \textit{Ans.}$$

Ex. 2. The base and altitude of a rectangle are 5 and 4 inches respectively. At a certain instant they are increasing continuously at the rate of 2 inches and 1 inch per second respectively. At what rate is the area of the rectangle increasing at that instant?

*Solution.* Let $x$ = base, $y$ = altitude; then $u = xy$ = area, $\frac{\partial u}{\partial x} = y$, $\frac{\partial u}{\partial y} = x$.

Substituting in (49),

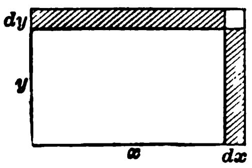
$$(A)\qquad \frac{du}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}.$$

But $x = 5$ in., $y = 4$ in., $\frac{dx}{dt} = 2$ in. per sec., $\frac{dy}{dt} = 1$ in. per sec.

$\therefore\ \frac{du}{dt} = (8 + 5)$ sq. in. per sec. $= 13$ sq. in. per sec. *Ans.*

NOTE. Considering $du$ as an infinitesimal increment of area due to the infinitesimal increments $dx$ and $dy$, $du$ is evidently the sum of two thin strips added on to the two sides. For, in $du = ydx + xdy$ (multiplying (*A*) by $dt$),

$ydx$ = area of vertical strip, and
$xdy$ = area of horizontal strip.



But the total increment $\Delta u$ due to the increments $dx$ and $dy$ is evidently

$$\Delta u = ydx + xdy + dxdy.$$

Hence the small rectangle in the upper right-hand corner ($=dxdy$) is evidently the difference between $\Delta u$ and $du$. This figure illustrates the fact that the total increment and the total differential of a function of several variables are not in general equal (see p. 141).

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[*] A geometric interpretation of this result will be given on p. 274.

**138. Differentiation of implicit functions.** The equation

$$(A)\qquad f(x, y) = 0$$

defines either $x$ or $y$ as an implicit function of the other.* It represents any equation containing $x$ and $y$ when all its terms have been transposed to the first member. Let

$$(B)\qquad u = f(x, y);$$

then
$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}. \qquad \text{(51), p. 199}$$

But from $(A)$, $f(x, y) = 0$. $\therefore u = 0$ and $\frac{du}{dx} = 0$; that is,

$$(C)\qquad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$,† we get

$$\textbf{(55)}\qquad \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}, \qquad \frac{\partial u}{\partial y} \neq 0$$

a formula for differentiating implicit functions. This formula in the form $(C)$ is equivalent to the process employed in § 75, pp. 83, 84, for differentiating implicit functions, and all the examples on p. 85 may be solved using formula **(55)**. Since

$$(D)\qquad f(x, y) = 0$$

for all admissible values of $x$ and $y$, we may say that **(55)** gives *the relative time rates of change of x and y which keep f(x, y) from changing at all.* Geometrically this means that the point $(x, y)$ must move on the curve whose equation is $(D)$, and **(55)** determines the direction of its motion at any instant. Since

$$u = f(x, y),$$

we may write **(55)** in the form

$$\textbf{(55 a)}\qquad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \qquad \frac{\partial f}{\partial y} \neq 0$$

---

* We assume that a small change in the value of $x$ causes only a small change in the value of $y$.

† It is assumed that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ exist.

**Ex.** 1. Given $x^2y^4 + \sin y = 0$; find $\frac{dy}{dx}$.

*Solution.* Let $f(x, y) = x^2y^4 + \sin y$.

$$\frac{\partial f}{\partial x} = 2xy^4, \ \frac{\partial f}{\partial y} = 4x^2y^3 + \cos y. \quad \therefore \text{from } (55\,a), \ \frac{dy}{dx} = -\frac{2xy^4}{4x^2y^3 + \cos y}. \quad \textit{Ans.}$$

**Ex. 2.** If $x$ increases at the rate of 2 inches per second as it passes through the value $x = 3$ inches, at what rate must $y$ change when $y = 1$ inch, in order that the function $2xy^2 - 3x^2y$ shall remain constant?

*Solution.* Let $f(x, y) = 2xy^2 - 3x^2y$; then

$$\frac{\partial f}{\partial x} = 2y^2 - 6xy, \ \frac{\partial f}{\partial y} = 4xy - 3x^2.$$

Substituting in (55 *a*),

$$\frac{dy}{dx} = -\frac{2y^2 - 6xy}{4xy - 3x^2}, \text{ or, } \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{2y^2 - 6xy}{4xy - 3x^2}. \qquad \text{By (34), p. 152}$$

But $x = 3$, $y = 1$, $\frac{dx}{dt} = 2$. $\therefore \frac{dy}{dt} = -2\frac{2}{15}$ ft. per second. *Ans.*

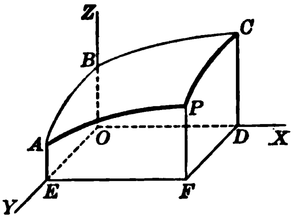
Let $P$ be the point $(x, y, z)$ on the surface given by the equation

$$(E) \qquad u = F(x, y, z) = 0,$$

and let $PC$ and $AP$ be sections made by planes through $P$ parallel to the $YOZ$ and $XOZ$ planes respectively. Along the curve $AP$, $y$ is constant, therefore from $(E)$, $z$ is an implicit function of $x$ alone, and we have from (55 *a*)

$$(56) \qquad \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}},$$

giving the slope at $P$ of the curve $AP$, § 133, p. 193.



$\frac{\partial z}{\partial x}$ is used instead of $\frac{dz}{dx}$ in the first member since $z$ was originally from $(E)$ an implicit function of $x$ and $y$, but (56) is deduced on the hypothesis that $y$ remains constant.

Similarly the slope at $P$ of the curve $PC$ is

$$(57) \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

## EXAMPLES

Find the total derivatives, using (49), (50), or (51) in the following six examples.

1. $u = z^2 + y^3 + zy,\ z = \sin x,\ y = e^x.$ *Ans.* $\frac{du}{dx} = 3e^{3x} + e^x(\sin x + \cos x) + \sin 2x.$

2. $u = \arctan(xy),\ y = e^x.$ *Ans.* $\frac{du}{dx} = \frac{e^x(1+x)}{1+x^2e^{2x}}.$

3. $u = \log(a^2 - \rho^2),\ \rho = a\sin\theta.$ *Ans.* $\frac{du}{d\theta} = -2\tan\theta.$

4. $u = v^2 + vy,\ v = \log s,\ y = e^s.$ *Ans.* $\frac{du}{ds} = \frac{2v+y}{s} + ve^s.$

5. $u = \arcsin(r - s),\ r = 3t,\ s = 4t^3.$ *Ans.* $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}.$

6. $u = \frac{e^{ax}(y - z)}{a^2 + 1},\ y = a\sin x,\ z = \cos x.$ *Ans.* $\frac{du}{dx} = e^{ax}\sin x.$

Using (53) or (54), find the total differentials in the next eight examples.

7. $u = by^2x + cx^3 + gy^3 + ex.$ *Ans.* $du = (by^2 + 2cx + e)\,dx + (2byx + 3gy^2)\,dy.$

8. $u = \log x^y.$ *Ans.* $du = \frac{y}{x}dx + \log x\,dy.$

9. $u = y^{\sin x}.$ *Ans.* $du = y^{\sin x}\log y\cos x\,dx + \frac{\sin x}{y \operatorname{covers} x}dy.$

10. $u = x^{\log y}.$ *Ans.* $du = u\left(\frac{\log y}{x}dx + \frac{\log x}{y}dy\right).$

11. $u = \frac{s+t}{s-t}.$ *Ans.* $du = \frac{2(s\,dt - t\,ds)}{(s-t)^2}.$

12. $u = \sin(pq).$ *Ans.* $du = \cos(pq)\,[q\,dp + p\,dq].$

13. $u = x^{yz}.$ *Ans.* $du = x^{yz-1}(yz\,dx + zx\log x\,dy + xy\log x\,dz).$

14. $u = \tan^2\phi\tan^2\theta\tan^2\psi.$ *Ans.* $du = 4u\left(\frac{d\phi}{\sin 2\phi} + \frac{d\theta}{\sin 2\theta} + \frac{d\psi}{\sin 2\psi}\right).$

15. Assuming the characteristic equation of a perfect gas to be

$$vp = Rt,$$

where $v$ = volume, $p$ = pressure, $t$ = absolute temperature, and $R$ a constant, what is the relation between the differentials $dv$, $dp$, $dt$? *Ans.* $v\,dp + p\,dv = R\,dt.$

16. Using the result in the last example as applied to air, suppose that in a given case we have found by actual experiment that

$$t = 300^\circ \text{ C.},\ p = 2000 \text{ lbs. per sq. ft.},\ v = 14.4 \text{ cubic feet.}$$

Find the change in $p$, assuming it to be uniform, when $t$ changes to 301° C., and $v$ to 14.5 cubic feet. $R = 96$. *Ans.* $-7.22$ lbs. per sq. ft.

In the remaining examples find $\frac{dy}{dx}$, using formula (~~56~~).(55)

17. $(x^2+y^2)^2 - a^2(x^2-y^2) = 0.$ *Ans.* $\frac{dy}{dx} = -\frac{x}{y}\cdot\frac{2(x^2+y^2)-a^2}{2(x^2+y^2)+a^2}.$

18. $e^y - e^x + xy = 0.$ *Ans.* $\frac{dy}{dx} = \frac{e^x - y}{e^y + x}.$

19. $\sin(xy) - e^{xy} - x^2y = 0.$ *Ans.* $\frac{dy}{dx} = \frac{y[\cos(xy) - e^{xy} - 2x]}{x[x + e^{xy} - \cos(xy)]}.$

20. $\sin x \sin y + \cos x \cos y - y = 0.$ *Ans.* $\frac{dy}{dx} = \frac{\cos x(\cos x + \sin y)}{1 - \sin x(\cos y - \sin y)}.$

### 139. Successive partial derivatives.

If

$$u = f(x, y),$$

then, in general,

$$\frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial y}$$

are functions of both $x$ and $y$, and may be differentiated again with respect to either independent variable, giving *successive partial derivatives.* Regarding $x$ alone as varying, we denote the results by

$$\frac{\partial^2 u}{\partial x^2},\ \frac{\partial^3 u}{\partial x^3},\ \frac{\partial^4 u}{\partial x^4},\ \cdots,\ \frac{\partial^n u}{\partial x^n},$$

or, when $y$ alone varies,

$$\frac{\partial^2 u}{\partial y^2},\ \frac{\partial^3 u}{\partial y^3},\ \frac{\partial^4 u}{\partial y^4},\ \cdots,\ \frac{\partial^n u}{\partial y^n},$$

the notation being similar to that employed for functions of a single variable.

If we differentiate $u$ with respect to $x$, regarding $y$ as constant, and then this result with respect to $y$, regarding $x$ as constant, we obtain

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \text{ which we denote by } \frac{\partial^2 u}{\partial y \partial x}.$$

Similarly, if we differentiate twice with respect to $x$ and then once with respect to $y$, the result is denoted by the symbol

$$\frac{\partial^3 u}{\partial y \partial x^2}.$$

**140. Order of differentiation immaterial.** Consider the function $f(x, y)$. Changing $x$ into $x + \Delta x$ and keeping $y$ constant, we get from the Theorem of Mean Value, (44), p. 168,

$$(A) \qquad f(x + \Delta x, y) - f(x, y) = \Delta x \cdot f_x(x + \theta_1 \cdot \Delta x, y). \quad 0 < \theta_1 < 1$$

[$a = x$, $\Delta a = \Delta x$, and since $x$ varies while $y$ remains constant, we get the partial derivative with respect to $x$.]

If we now change $y$ to $y + \Delta y$ and keep $x$ and $\Delta x$ constant, the total increment of the left-hand member of $(A)$ is

$$(B) \quad [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] - [f(x + \Delta x, y) - f(x, y)].$$

The total increment of the right-hand member of $(A)$ found by the Theorem of Mean Value, (44), p. 168, is

$$\begin{aligned}(C) \quad & \Delta x f_x'(x + \theta_1 \cdot \Delta x, y + \Delta y) - \Delta x f_x'(x + \theta_1 \cdot \Delta x, y) \\ & = \Delta y \Delta x f_{yx}''(x + \theta_1 \cdot \Delta x, y + \theta_2 \cdot \Delta y). \quad 0 < \theta_2 < 1\end{aligned}$$

[$a = y$, $\Delta a = \Delta y$, and since $y$ varies while $x$ and $\Delta x$ remain constant, we get the partial derivative with respect to $y$.]

Since the increments $(B)$ and $(C)$ must be equal,

$$\begin{aligned}(D) \quad & [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] - [f(x + \Delta x, y) - f(x, y)] \\ & = \Delta y \Delta x f_{yx}''(x + \theta_1 \cdot \Delta x, y + \theta_2 \cdot \Delta y).\end{aligned}$$

In the same manner, if we take the increments in the reverse order,

$$\begin{aligned}(E) \quad & [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)] - [f(x, y + \Delta y) - f(x, y)] \\ & = \Delta x \Delta y f_{xy}''(x + \theta_3 \cdot \Delta x, y + \theta_4 \cdot \Delta y),\end{aligned}$$

$\theta_3$ and $\theta_4$ also lying between zero and unity.

The left-hand members of $(D)$ and $(E)$ being identical, we have

$$(F) \quad f_{yx}''(x + \theta_1 \cdot \Delta x, y + \theta_2 \cdot \Delta y) = f_{xy}''(x + \theta_3 \cdot \Delta x, y + \theta_4 \cdot \Delta y).$$

Taking the limit of both sides as $\Delta x$ and $\Delta y$ approach zero as limits, we have *

$$(G) \qquad f_{xy}''(x, y) = f_{xy}''(x, y),$$

since these functions are assumed continuous. Placing

$$u = f(x, y),$$

$(G)$ may be written

$$(58) \qquad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

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* Assuming the continuity of the first partial derivatives and the existence and continuity of $f_{xy}''$ and $f_{yx}''$.

That is, *the operations of differentiating with respect to x and with respect to y are commutative.*

This may be easily extended to higher derivatives. For instance, since (58) is true,

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^3 u}{\partial y \partial x^2}.$$

Similarly for functions of three or more variables.

Ex. 1. Given $u = x^3y - 3x^2y^3$; verify $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

*Solution.* $\frac{\partial u}{\partial x} = 3x^2y - 6xy^3$, $\frac{\partial^2 u}{\partial y \partial x} = 3x^2 - 18xy^2$,

$\frac{\partial u}{\partial y} = x^3 - 9x^2y^2$, $\frac{\partial^2 u}{\partial x \partial y} = 3x^2 - 18xy^2$; hence verified.

### EXAMPLES

1. $u = \cos(x + y)$; verify $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

2. $u = \frac{y^2 + x^2}{y^2 - x^2}$; verify $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

3. $u = y \log(1 + xy)$; verify $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

4. $u = \text{arc} \tan \frac{r}{s}$; verify $\frac{\partial^3 u}{\partial r^2 \partial s} = \frac{\partial^3 u}{\partial s \partial r^2}$.

5. $u = \sin(\theta^2 \phi)$; verify $\frac{\partial^3 u}{\partial \theta \partial \phi^2} = \frac{\partial^3 u}{\partial \phi^2 \partial \theta}$.

6. $u = 6e^xy^2z + 3e^yx^2z^2 + 2e^zx^3y - xyz$; show that $\frac{\partial^4 u}{\partial x^2 \partial y \partial z} = 12(e^xy + e^yz + e^zx)$.

7. $u = e^{xyz}$; show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)u$.

8. $u = \frac{x^2y^2}{x + y}$; show that $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial x}$.

9. $u = (x^2 + y^2)^{\frac{1}{3}}$; show that $3x\frac{\partial^2 u}{\partial x \partial y} + 3y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$.

10. $u = y^2z^2e^{\frac{x}{2}} + z^2x^2e^{\frac{y}{2}} + x^2y^2e^{\frac{z}{2}}$; show that $\frac{\partial^6 u}{\partial x^2 \partial y^2 \partial z^2} = e^{\frac{x}{2}} + e^{\frac{y}{2}} + e^{\frac{z}{2}}$.

11. $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$; show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

12. $u = \sec(p + aq) + \tan(q - ap)$; prove $\frac{\partial^2 u}{\partial q^2} = a^2\frac{\partial^2 u}{\partial p^2}$.

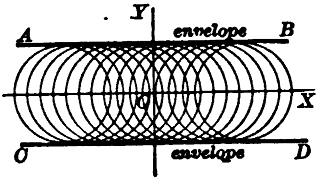
# CHAPTER XVIII

## ENVELOPES

**141. Family of curves. Variable parameter.** The equation of a curve generally involves, besides the variables $x$ and $y$, certain constants upon which the size, shape, and position of that particular curve depend. For example, the locus of the equation

$$(A) \qquad (x-a)^2 + y^2 = r^2$$

is a circle whose center lies on the axis of $X$ at a distance of $a$ from the origin, its size depending on the radius $r$. Suppose $a$ to take on a series of values, then we shall have a corresponding series of circles differing in their distances from the origin, as shown in the figure.



Any system of curves formed in this way is called a *family of curves*, and the quantity $a$, which is constant for any one curve, but changes in passing from one curve to another, is called a *variable parameter.*

As will appear later on, problems occur which involve two or more parameters. The above series of circles is said to be a *family depending on one parameter.* To indicate that $a$ enters as a variable parameter it is usual to insert it in the functional symbol, thus:

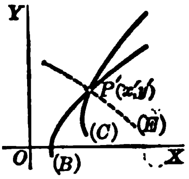
$$f(x, y, a) = 0.$$

**142. Envelope of a family of curves depending on one parameter.** Any two neighboring curves of a family will in general intersect.* If the corresponding values of the parameter are $a$ and $a + \Delta a$, the point or points of intersection, if these exist, will in general tend to definite limiting positions (points) as $\Delta a$ approaches zero. The locus of all such limiting points is called the *envelope of the family*

---

* An exception to this would be the system of concentric circles we get from (A) when $a$ is constant and $r$ varies, no two of which would intersect.

*of curves.* Thus, in the last figure, the limiting positions of the points of intersection of the circles are all on the straight lines $AB$ and $CD$, which form therefore the envelope of the family of circles



Now find *the equation of the envelope of a family of curves depending on one parameter.* Let

$$(B) \qquad f(x, y, a) = 0 \text{ and}$$

$$(C) \qquad f(x, y, a + \Delta a) = 0$$

be two neighboring curves of the same family intersecting at a point $(x', y')$; and let us find the limiting position of this point of intersection as $\Delta a$ approaches the limit zero.

We can find the equation of a third curve through $(x', y')$ by applying the Theorem of Mean Value, (44), p. 168, to $(B)$ and $(C)$, regarding $a$ as the variable and $x$ and $y$ as constants. For we have

$$(D) \quad f(x, y, a + \Delta a) - f(x, y, a) = \Delta a f_a'(x, y, a + \theta \cdot \Delta a). \quad 0 < \theta < 1$$

Since $P'$ lies on both of the curves $(B)$ and $(C)$, the left-hand members of their equations vanish for $x = x'$ and $y = y'$. Hence the left-hand member of $(D)$ must vanish for the same values, and consequently the right-hand member also. Therefore

$$(E) \qquad f_a'(x, y, a + \theta \cdot \Delta a) = 0$$

is the equation of a third curve passing through the intersection of $(B)$ and $(C)$. If then $(B)$ and $(C)$ intersect in a point which approaches a fixed point as a limit as $\Delta a$ approaches zero, we get in general

$$(F) \qquad f_a'(x, y, a) = 0$$

as the equation of a curve which passes through the limit of the intersection of $(B)$ and $(C)$. In general $(F)$ is distinct from $(B)$ and therefore has a definite intersection with it.

Since the coördinates of the points on the envelope satisfy both $(F)$ and $(B)$, its equation is found by eliminating $a$ between these equations. The equation of the envelope is therefore a new relation between $x$ and $y$ that is independent of $a$.*

---

* By definition we should solve $(B)$ and $(C)$ simultaneously for their point of intersection and then pass to the limit. In practice, however, it is found to be more convenient first to pass to the limit and then solve for $x$ and $y$, just as we do here. It is not self-evident by any means that these two processes give the same results in all cases, but it is a fact that the results are identical in all the applications made in this book.

On account of the process of elimination that is involved, no detailed method of procedure can be given for finding the envelope that will apply in all cases. In a large number of problems, however, the student may be guided by the following

**General directions for finding the envelope.**

**First step.** ***Differentiate with respect to the variable parameter, considering all other quantities involved in the given equation as constants.***

**Second step.** *Solve the result for the variable parameter.*

**Third step.** *Substitute this value of the variable parameter in the given equation. This gives the equation of the envelope.*

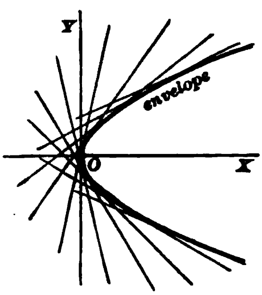
**Ex. 1.** Find the envelope of the straight line $y = mx + \frac{p}{m}$, where the slope $m$ is the variable parameter.

*Solution.* $y = mx + \frac{p}{m}$.

*First step.* $o = x - \frac{p}{m^2}$.

*Second step.* $m = \pm \sqrt{\frac{p}{x}}$.

*Third step.* $y = \pm \sqrt{\frac{p}{x}} \cdot x \pm \sqrt{\frac{x}{p}} \cdot p = \pm 2 \sqrt{px}$,



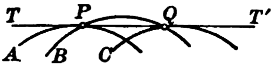
and squaring, $y^2 = 4px$, a parabola, is the equation of the envelope. The family of straight lines formed by varying the slope $m$ is shown in the figure, each line being tangent to the envelope, for we know from Analytic Geometry that $y = mx + \frac{p}{m}$ is the tangent to the parabola $y^2 = 4px$ expressed in terms of its own slope $m$.

**143. The envelope touches each curve of the family at the limiting points on that curve.**

*Geometrical proof.* Let $A$, $B$, $C$ be three neighboring curves of the family, $A$ and $B$ intersecting at $P$, and $B$ and $C$ at $Q$. Draw $TT'$ through $P$ and $Q$. Now let $A$ and $C$ approach coincidence with $B$, that is, let $A$, $B$, $C$ become consecutive curves[*] of the family. Then $TT'$ becomes a tangent to $B$, having two consecutive points $P$ and $Q$ in common with it. But then $P$ and $Q$ will also become consecutive points of the envelope by definition; hence $TT'$ will at the same time become a tangent to the envelope. Therefore $B$ and the envelope have a common tangent; similarly for every curve of the family. Thus in the example of the last section we noticed that the parabola had the family of straight lines as tangents.

---

[*] The limiting position of any point of intersection (as $P$ in figure, p. 211) is sometimes called the point of intersection of two *consecutive* curves of the family. Similarly the line which $TT'$ approaches as $P$ approaches $Q$, i.e. the tangent to $B$ at $Q$, is said to pass through two *consecutive* points of the curve. Of course there is no one curve that is consecutive to another nor any one point that is consecutive to another in the ordinary sense of the word, but geometrical considerations have suggested the above phraseology and it is understood to be merely a brief way of indicating the actual condition of affairs as stated in the definitions.



*Analytical proof.* Consider the family of curves represented by

(A) $$f(x, y, a) = 0.$$

The slope at any point on (A) is the value of $\frac{dy}{dx}$ from

(B) $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0;$$ (51), p. 199

where, in differentiating, $a$ must be kept constant.

From the previous section, we know that the envelope of the family of curves is found by eliminating $a$ between (A) and

(C) $$\frac{\partial}{\partial a} f(x, y, a) = 0.$$

If we suppose (C) solved for $a$ in terms of $x$ and $y$ and the result substituted in (A), it is evident that equation (A) would then be the equation of the envelope. Hence the slope of the envelope may be found by taking the *total derivative* of (A), (52), p. 199, regarding $a$ as a certain function of $x$ and $y$ determined by (C). This gives

(D) $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial a}\frac{da}{dx} = 0.$$

Suppose now that the coördinates of the point $(x, y)$ satisfy both (A) and (C); that point is therefore on the curve (A) and also on the envelope; and, by (C), the last term in (D) vanishes, reducing (D) to the same form as (B). Hence at the point $(x, y)$ the slope is the same for the curve (A) and the envelope, so that a limiting point of intersection on any member of the family is a point of contact of this curve with the envelope.*

**144. Parametric equations of the envelope of a family depending on one parameter.** Instead of finding the equation of the envelope in rectangular form by the method of § 142, p. 210, it is sometimes more convenient to get the equations of the envelope in parametric form by solving

$$f(x, y, a) = 0 \text{ and } \frac{\partial}{\partial a} f(x, y, a) = 0$$

for $x$ and $y$ in terms of $a$. Thus:

---

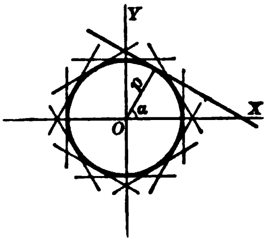
* In the special case when $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ or $\frac{\partial f}{\partial y} = 0$ for all points of our locus this reasoning fails.

**Ex. 1.** Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = p$, $\alpha$ being the variable parameter.

*Solution.* $$(A) \qquad x\cos\alpha + y\sin\alpha = p.$$

Differentiating $(A)$ with respect to $\alpha$,

$$(B) \qquad -x\sin\alpha + y\cos\alpha = 0.$$



Multiplying $(A)$ by $\cos\alpha$ and $(B)$ by $\sin\alpha$ and subtracting, we get

$$x = p\cos\alpha.$$

Similarly, eliminating $x$ between $(A)$ and $(B)$, we get

$$y = p\sin\alpha.$$

The *parametric equations of the envelope* are therefore

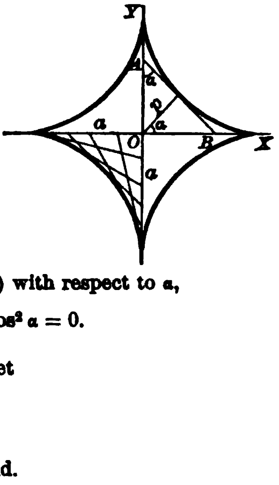
$$(C) \qquad \begin{cases} x = p\cos\alpha, \\ y = p\sin\alpha; \end{cases}$$

$\alpha$ being the parameter. Squaring equations $(C)$ and adding, we get

$$x^2 + y^2 = p^2,$$

the rectangular equation of the envelope, which is a circle.

**Ex. 2.** Find the envelope of a line of constant length $a$, whose extremities move along two fixed rectangular axes.



*Solution.* Let $AB = a$ in length, and let

$$(A) \qquad x\cos\alpha + y\sin\alpha - p = 0$$

be its equation. Now as $AB$ moves always touching the two axes, both $\alpha$ and $p$ will vary. But $p$ may be found in terms of $\alpha$. For, $AO = AB\cos\alpha = a\cos\alpha$, and $p = AO\sin\alpha = a\sin\alpha\cos\alpha$. Substituting in $(A)$,

$$(B) \qquad x\cos\alpha + y\sin\alpha - a\sin\alpha\cos\alpha = 0,$$

where $\alpha$ is the variable parameter. Differentiating $(B)$ with respect to $\alpha$,

$$(C) \qquad -x\sin\alpha + y\cos\alpha + a\sin^2\alpha - a\cos^2\alpha = 0.$$

Solving $(B)$ and $(C)$ for $x$ and $y$ in terms of $\alpha$, we get

$$(D) \qquad \begin{cases} x = a\sin^3\alpha, \\ y = a\cos^3\alpha, \end{cases}$$

the parametric equations of the envelope, a hypocycloid.

The corresponding rectangular equation is found from equations $(D)$ by eliminating $\alpha$ as follows:

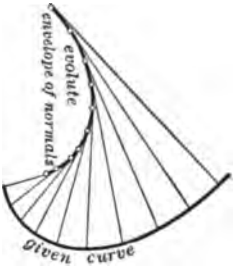
$$x^{\frac{2}{3}} = a^{\frac{2}{3}}\sin^2\alpha.$$

$$y^{\frac{2}{3}} = a^{\frac{2}{3}}\cos^2\alpha.$$

Adding, $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

the rectangular equation of the hypocycloid.

**145. The evolute of a given curve considered as the envelope of its normals.** Since the normals to a curve are all tangent to the evolute, § 129, p. 186, it is evident that *the evolute of a curve may also be defined as the envelope of its normals;* that is, as the locus of the ultimate intersections of neighboring normals. It is also interesting to notice that if we find the parametric equations of the envelope by the method of the previous section, we get the coördinates $x$ and $y$ of the center of curvature; so that we have here *a second method for finding the coördinates of the center of curvature.* If we then eliminate the variable parameter, we have a relation between $x$ and $y$ which is the rectangular equation of the evolute (envelope of the normals).



Ex. 1. **Find the evolute of the parabola $y^2 = 4px$ considered as the envelope of its normals.**

*Solution.* The equation of the normal at any point $(x', y')$ is

$$y - y' = -\frac{y'}{2p}(x - x')$$

from (2), p. 90. As we are considering the normals all along the curve, both $x'$ and $y'$ will vary. Eliminating $x'$ by means of $y'^2 = 4px'$, we get the equation of the normal to be

$$(A) \qquad y - y' = \frac{y'^3}{8p^2} - \frac{xy'}{2p}.$$

Considering $y'$ as the variable parameter, we wish to find the envelope of this family of normals. Differentiating $(A)$ with respect to $y'$,

$$-1 = \frac{3y'^2}{8p^2} - \frac{x}{2p}$$

and solving for $x$,

$$(B) \qquad x = \frac{3y'^2 + 8p^2}{4p}.$$

Substituting this value of $x$ in $(A)$ and solving for $y$,

$$(C) \qquad y = -\frac{y'^3}{4p^2}.$$

$(B)$ and $(C)$ are then the coördinates of the center of curvature of the parabola. Taken together, $(B)$ and $(C)$ are the parametric equations of the evolute in terms of the parameter $y'$. Eliminating $y'$ between $(B)$ and $(C)$ gives

$$27py^2 = 4(x - 2p)^3,$$

the rectangular equation of the evolute of the parabola. This is the same result we obtained in Ex. 1, p. 188, by the first method.

**146. Two parameters connected by one equation of condition.** Many problems occur where it is convenient to use two parameters connected by an equation of condition. For instance, the example given in the last section involves the two parameters $x'$ and $y'$ which are connected by the equation of the curve. In this case we eliminated $x'$, leaving only the one parameter $y'$.

However, when the elimination is difficult to perform, both the given equation and the equation of condition between the two parameters may be differentiated with respect to one of the parameters, regarding either parameter as a function of the other. By studying the solution of the following problem the process will be made clear.

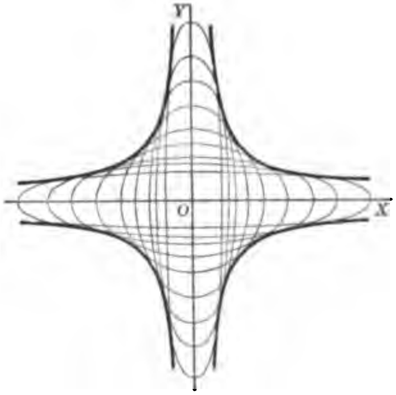
Ex. 1. **Find the envelope of the family of ellipses whose axes coincide and whose area is constant.**

*Solution.* (*A*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is the equation of the ellipse where $a$ and $b$ are the variable parameters connected by the equation

(*B*) $\pi ab = k,$

$\pi ab$ being the area of an ellipse whose semiaxes are $a$ and $b$. Differentiating (*A*) and (*B*), regarding $a$ and $b$ as variables and $x$ and $y$ as constants, we have, using differentials,



$$\frac{x^2da}{a^3} + \frac{y^2db}{b^3} = 0, \text{ from } (A),$$

and $bda + adb = 0$, from (*B*).

Transposing one term in each to the second member and dividing, we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}.$$

Therefore, from (*A*), $\frac{x^2}{a^2} = \frac{1}{2}$ and $\frac{y^2}{b^2} = \frac{1}{2}$,

giving $a = \pm x\sqrt{2}$ and $b = \pm y\sqrt{2}$.

Substituting these values in (*B*), we get the envelope

$$xy = \pm \frac{k}{2\pi},$$

a pair of conjugate rectangular hyperbolas (see figure).

### EXAMPLES

1. Find the envelope of the family of straight lines $y = 2mx + m^4$, $m$ being the variable parameter. *Ans.* $x = -2m^3$, $y = -3m^4$; or, $16y^3 + 27x^4 = 0.$*

2. Find the envelope of the family of parabolas $y^2 = a(x - a)$, $a$ being the variable parameter. *Ans.* $x = 2a$, $y = \pm a$; or, $y = \pm \frac{1}{2}x$.

3. Find the envelope of the family of circles $x^2 + (y - \beta)^2 = r^2$, $\beta$ being the variable parameter. *Ans.* $x = \pm r$.

4. Find the equation of the curve having as tangents the family of straight lines $y = mx \pm \sqrt{a^2m^2 + b^2}$, the slope $m$ being the variable parameter.
*Ans.* The ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

5. Find the envelope of the family of circles whose diameters are double ordinates of the parabola $y^2 = 4px$. *Ans.* The parabola $y^2 = 4p(p + x)$.

6. Find the envelope of the family of circles whose diameters are double ordinates of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. *Ans.* The ellipse $\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1$.

7. A circle moves with its center on the parabola $y^2 = 4ax$, and its circumference passes through the vertex of the parabola. Find the equation of the locus of the points of ultimate intersection of the circles.
*Ans.* The cissoid $y^2(x + 2a) + x^3 = 0$.

8. Find the curve whose tangents are $y = lx \pm \sqrt{al^2 + bl + c}$, the slope $l$ being supposed to vary. *Ans.* $4(ay^2 + bxy + cx^2) = 4ac - b^2$.

9. Find the evolute of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, taking the equation of normal in the form

$$by = ax \tan \phi - (a^2 - b^2) \sin \phi,$$

the eccentric angle $\phi$ being the parameter.

*Ans.* $x = \frac{a^2 - b^2}{a} \cos^3 \phi$, $y = \frac{b^2 - a^2}{b} \sin^3 \phi$; or, $(ax)^{\frac{2}{3}} + (ay)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

10. Find the evolute of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the equation of whose normal is

$$y \cos \tau - x \sin \tau = a \cos 2\tau,$$

$\tau$ being the parameter. *Ans.* $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

11. Find the envelope of the circles which pass through the origin and have their centers on the hyperbola $x^2 - y^2 = c^2$.
*Ans.* The lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

12. Find the envelope of a line such that the sum of its intercepts on the axes equals $c$. *Ans.* The parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = c^{\frac{1}{2}}$.

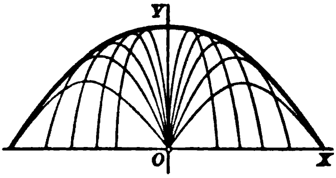
---

* When two answers are given, the first is in parametric form and the second in rectangular form.

13. Find the envelope of the family of ellipses $b^2x^2 + a^2y^2 = a^2b^2$, when the sum of its semiaxes equals $c$. *Ans.* The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.

14. Find the envelope of the ellipses whose axes coincide, and such that the distance between the extremities of the major and minor axes is constant and equal to $l$. *Ans.* A square whose sides are $(x \pm y)^2 = l^2$.

15. Projectiles are fired from a gun with an initial velocity $v_0$. Supposing the gun can be given any elevation and is kept always in the same vertical plane, what is the envelope of all possible trajectories, the resistance of the air being neglected?



*Hint.* The equation of any trajectory is

$$y = x \tan \alpha - \frac{gx^2}{2v_0 \cos^2 \alpha},$$

$\alpha$ being the variable parameter.

*Ans.* The parabola $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$.

# CHAPTER XIX

## SERIES

**147. Introduction.** A *series* is a succession of separate numbers which is formed according to some rule or law. Each number is called a term of the series. Thus,

$$1, 2, 4, 8, \cdots, 2^{n-1}$$

is a series whose law of formation is that each term after the first is found by multiplying the preceding term by 2; hence we may write down as many more terms of the series as we please, and any particular term of the series may be found by substituting *the number of that term in the series* for $n$ in the expression $2^{n-1}$, which is called the *general* or $n$th *term* of the series.

### EXAMPLES

In the following six series:

(a) Discover by inspection the law of formation;
(b) write down several terms more in each;
(c) find the $n$th or *general term*.

| | *Series* | *nth term* |
|---|---|---|
| 1. | $1, 3, 9, 27, \cdots.$ | $3^{n-1}.$ |
| 2. | $-a, +a^2, -a^3, +a^4, \cdots.$ | $(-a)^n.$ |
| 3. | $1, 4, 9, 16, \cdots.$ | $n^2.$ |
| 4. | $x, \frac{x^2}{2}, \frac{x^3}{3}, \frac{x^4}{4}, \cdots.$ | $\frac{x^n}{n}.$ |
| 5. | $4, -2, +1, -\frac{1}{2}, \cdots.$ | $4(-\frac{1}{2})^{n-1}.$ |
| 6. | $\frac{3y}{2}, \frac{5y^2}{5}, \frac{7y^3}{10}, \cdots.$ | $\frac{2n+1}{n^2+1}y^n.$ |

Write down the first four terms of each series whose $n$th or *general term* is given below.

| | *nth term* | *Series* |
|---|---|---|
| 7. | $n^2x^n.$ | $x, 4x^2, 9x^3, 16x^4.$ |
| 8. | $\frac{x^n}{1+\sqrt{n}}.$ | $\frac{x}{2}, \frac{x^2}{1+\sqrt{2}}, \frac{x^3}{1+\sqrt{3}}, \frac{x^4}{1+\sqrt{4}}.$ |

| | *nth term* | *Series* |
|---|---|---|
| 9. | $\dfrac{n+2}{n^3+1}$. | $\dfrac{3}{2}, \dfrac{4}{9}, \dfrac{5}{10}, \dfrac{6}{17}$. |
| 10. | $\dfrac{n}{2^n}$. | $\dfrac{1}{2}, \dfrac{2}{4}, \dfrac{3}{8}, \dfrac{4}{16}$. |
| 11. | $\dfrac{(\log a)^n x^n}{\underline{\vert n}}$. | $\dfrac{\log a\cdot x}{1}, \dfrac{\log^2 a\cdot x^2}{2}, \dfrac{\log^3 a\cdot x^3}{6}, \dfrac{\log^4 a\cdot x^4}{24}$. |
| 12. | $\dfrac{(-1)^{n-1}x^{2n-2}}{\underline{\vert 2n-1}}$. | $\dfrac{1}{1}, -\dfrac{x^2}{\underline{\vert 3}}, \dfrac{x^4}{\underline{\vert 5}}, -\dfrac{x^6}{\underline{\vert 7}}$. |

**148. Infinite series.** Consider the series of $n$ terms

$$(A)\qquad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, \frac{1}{2^{n-1}};$$

and let $S_n$ denote the sum of the series. Then

$$(B)\qquad S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}.$$

Evidently $S_n$ is a function of $n$, for

when $n = 1,\ S_1 = 1 \qquad = 1,$

when $n = 2,\ S_2 = 1 + \frac{1}{2} \qquad = 1\frac{1}{2},$

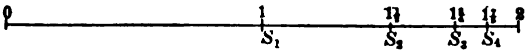
when $n = 3,\ S_3 = 1 + \frac{1}{2} + \frac{1}{4} \qquad = 1\frac{3}{4},$

when $n = 4,\ S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad = 1\frac{7}{8},$

. . . . . . . . . . . . .

when $n = n,\ S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} \qquad = 2 - \frac{1}{2^{n-1}}.$*

Mark off points on a straight line whose distances from a fixed point 0 correspond to these different sums. It is seen that the



point corresponding to any sum bisects the distance between the preceding point and 2. Hence it appears geometrically that when $n$ increases without limit

$$\text{limit } S_n = 2.$$

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* Found by 6, p. 1, for the sum of a geometric series.

We also see that this is so from arithmetical considerations, for

$$\lim_{n=\infty} S_n = \lim_{n=\infty}\left(2 - \frac{1}{2^{n-1}}\right) = 2.^*$$

**[Since when $n$ increases without limit, $\frac{1}{2^{n-1}}$ approaches zero as a limit.]**

We have so far discussed only a particular series ($A$) when the number of terms increases without limit. Let us now consider the general problem, using the series

$$(C) \qquad u_1,\ u_2,\ u_3,\ u_4,\ \cdots,$$

whose terms may be either positive or negative. Denoting by $S_n$ the sum of the first $n$ terms, we have

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n,$$

and $S_n$ is a function of $n$. If we now let the number of terms ($= n$) increase without limit, one of two things may happen: either

CASE I. **$S_n$ approaches a limit,** say $u$, indicated by

$$\lim_{n=\infty} S_n = u\text{; or,}$$

CASE II. **$S_n$ approaches no limit.**

In either case ($C$) is called an *infinite series*. In Case I the infinite series is said to be *convergent* and *to converge to the value u*, or *to have the value u*, or *to have the sum u*. The infinite geometric series discussed at the beginning of this section is an example of a convergent series, and it converges to the value 2. In fact, the simplest example of a convergent series is the infinite geometric series

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \cdots,$$

where $r$ is numerically less than unity. The sum of the first $n$ terms of this series is, by 6, p. 1,

$$S_n = \frac{a(1+r^n)}{1-r} = \frac{a}{1-r} + \frac{ar^n}{1-r}.$$

If we now suppose $n$ to increase without limit, the first fraction

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[*] Such a result is sometimes, for the sake of brevity, called the *sum* of the series; but the student must not forget that 2 is *not* the sum but the *limit of the sum*, as the number of terms increases without limit.

on the right-hand side remains unchanged, while the second approaches zero as a limit. Hence

$$\lim_{n=\infty} S_n = \frac{a}{1-r},$$

a *perfectly definite number* in any given case.

In Case II the infinite series is said to be *nonconvergent*.[*] Series under this head may be divided into two classes.

*First class. Divergent series*, in which the sum of $n$ terms increases indefinitely in numerical value as $n$ increases without limit; for example, the series from which we get

$$S_n = 1 + 2 + 3 + \cdots + n.$$

As $n$ increases without limit, $S_n$ increases without limit and therefore the series is *divergent.*

*Second class. Oscillating series*, of which

$$S_n = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$$

is an example. Here $S_n$ is zero or unity according as $n$ is even or odd, and although $S_n$ does not become infinite as $n$ increases without limit, it does not tend to a limit, but oscillates. It is evident that if all the terms of a series have the same sign the series cannot oscillate.

Since the sum of a converging series is a perfectly definite number, while such a thing as the sum of a nonconvergent series does not exist, it follows at once that it is absolutely essential in any given problem involving infinite series to determine whether or not the series is convergent. This is often a problem of great difficulty, and we shall consider only the simplest cases.

**149. Existence of a limit.** When a series is given we cannot in general, as in the case of a geometric series, actually find the number which is the limit of $S_n$. But although we may not know how to compute the numerical value of that limit, it is of prime importance to know that a *limit does exist*, for otherwise the series may be nonconvergent. When examining a series to determine whether or not it is convergent, the following theorems, which we state without proofs, are found to be of fundamental importance.[†]

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[*] Some writers use *divergent* as equivalent to *nonconvergent*.

[†] See Osgood's *Introduction to Infinite Series*, pp. 4, 14, 64.

**Theorem I.** *If $S_n$ is a variable that always increases as n increases, but always remains less than some definite fixed number A, then as n increases without limit, $S_n$ will approach a definite limit which is not greater than A.*

**Theorem II.** *If $S_n$ is a variable that always decreases as n increases, but always remains greater than some definite fixed number B, then as n increases without limit, $S_n$ will approach a definite limit which is not less than B.*

**Theorem III.** *The necessary and sufficient condition that $S_n$ shall approach some definite fixed number as a limit as n increases without limit is that*

$$\lim_{n=\infty}(S_{n+p}-S_n)=0$$

*for all values of the integer p.*

**150. Fundamental test for convergence.** Summing up first $n$ and then $n+p$ terms of a series, we have

$$(A)\qquad S_n=u_1+u_2+u_3+\cdots+u_n.$$

$$(B)\qquad S_{n+p}=u_1+u_2+u_3+\cdots+u_n+u_{n+1}+\cdots+u_{n+p}.$$

Subtracting $(A)$ from $(B)$,

$$(C)\qquad S_{n+p}-S_n=u_{n+1}+u_{n+2}+\cdots+u_{n+p}.$$

From Theorem III we know that the *necessary* and *sufficient* condition that the series shall be convergent is that

$$\lim_{n=\infty}(S_{n+p}-S_n)=0$$

for every value of $p$. But this is the same as the left-hand member of $(C)$; therefore from the right-hand member the condition may also be written

$$(D)\qquad \lim_{n=\infty}(u_{n+1}+u_{n+2}+\cdots+u_{n+p})=0.$$

Since $(D)$ is true for every value of $p$, then letting $p=1$, a *necessary* condition for convergence is that

$$\lim_{n=\infty}(u_{n+1})=0;$$

or, what amounts to the same thing,

$$(E)\qquad \lim_{n=\infty}(u_n)=0.$$

Hence, if the general (or $n$th) term of a series does not approach zero as $n$ approaches infinity, we know at once that the series is nonconvergent and we need proceed no further. However, $(E)$ is not a *sufficient* condition, that is, even if the $n$th term does approach zero we cannot state positively that the series is convergent; for, consider the harmonic series

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \cdots,\ \frac{1}{n}.$$

Here $$\underset{n=\infty}{\text{limit}}(u_n)=\underset{n=\infty}{\text{limit}}\left(\frac{1}{n}\right)=0,$$

that is, condition $(E)$ is fulfilled. Yet we may show that the harmonic series is not convergent by the following comparison:

$$(F)\qquad 1+\tfrac{1}{2}+[\tfrac{1}{3}+\tfrac{1}{4}]+[\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}]+[\tfrac{1}{9}+\cdots+\tfrac{1}{16}]+\cdots.$$

$$(G)\qquad \tfrac{1}{2}+\tfrac{1}{2}+[\tfrac{1}{4}+\tfrac{1}{4}]+[\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}]+[\tfrac{1}{16}+\cdots+\tfrac{1}{16}]+\cdots.$$

We notice that every term of $(G)$ is equal to or less than the corresponding term of $(F)$, so that the sum of any number of the first terms of $(F)$ will be greater than the sum of the corresponding terms of $(G)$. But since the sum of the terms grouped in each bracket in $(G)$ equals $\frac{1}{2}$, the sum of $(G)$ may be made as large as we please by taking terms enough. The sum $(G)$ increases indefinitely as the number of terms increases without limit; hence $(G)$, and therefore also $(F)$, is divergent.

We shall now proceed to deduce special tests which as a rule are easier to apply than the above theorems.

**151. Comparison test for convergence.** In many cases, an example of which was given in the last section, it is easy to determine whether or not a given series is convergent by comparing it term by term with another series whose character is known. *Let*

$$(A)\qquad u_1+u_2+u_3+\cdots$$

*be a series of positive terms which it is desired to test for convergence. If a series of positive terms already known to be convergent, namely,*

$$(B)\qquad a_1+a_2+a_3+\cdots,$$

*can be found whose terms are never less than the corresponding terms* *in the series (A) to be tested, then (B) is a convergent series and its sum does not exceed that of (A).*

*Proof.* Let $$s_n = u_1 + u_2 + u_3 + \cdots + u_n,$$
and $$S_n = a_1 + a_2 + a_3 + \cdots + a_n;$$
and suppose that $$\lim_{n=\infty} S_n = A.$$

Then, since $$S_n < A \text{ and } s_n \leqq S_n,$$
it follows that $s_n < A$. Hence, by Theorem I, p. 221, $s_n$ approaches a limit; therefore the series $(A)$ is convergent and the limit of its sum is not greater than $A$.

**Ex. 1.** Test the series

$$(C) \qquad 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots.$$

*Solution.* Each term after the first is less than the corresponding term of the geometric series

$$(D) \qquad 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots,$$

which is known to be convergent (p. 218); hence $(C)$ is also convergent.

Following a line of reasoning similar to that applied to $(A)$ and $(B)$, it is evident that, if

$$(E) \qquad u_1 + u_2 + u_3 + \cdots$$

*is a series of positive terms to be tested which are never less than the corresponding terms of the series of positive terms, namely,*

$$(F) \qquad b_1 + b_2 + b_3 + \cdots,$$

*known to be divergent, then (E) is a divergent series.*

Ex. 2. Test the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots.$$

*Solution.* This series is divergent since its terms are greater than the corresponding terms of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots,$$

which is known (p. 222) to be divergent.

Ex. 3. Test the following series for different values of $p$.

$$(G) \qquad 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots.$$

*Solution.* Grouping the terms, we have, when $p > 1$,

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}},$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \left(\frac{1}{2^{p-1}}\right)^2,$$

$$\frac{1}{8^p} + \cdots + \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} = \frac{8}{8^p} = \left(\frac{1}{2^{p-1}}\right)^3,$$

and so on. Construct the series

$$(H) \qquad 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots.$$

*When* $p > 1$, series $(H)$ is a geometric series with the common ratio less than unity, and is therefore convergent. But the sum of $(G)$ is less than the sum of $(H)$, as shown by the above inequalities; therefore $(G)$ is also convergent.

*When* $p = 1$, series $(G)$ becomes the harmonic series which we saw was divergent, and neither of the above tests applies.

*When* $p < 1$, the terms of series $(G)$ will, after the first, be greater than the corresponding terms of the harmonic series; hence $(G)$ is divergent.

**152. Cauchy's ratio test for convergence.** Let

$$(A) \qquad u_1 + u_2 + u_3 + \cdots$$

be a series of positive terms to be tested.

Divide any general term by the one that immediately precedes it, i.e. form the test ratio $\frac{u_{n+1}}{u_n}$

As $n$ increases without limit, let $\underset{n=\infty}{\text{limit}} \frac{u_{n+1}}{u_n} = \rho$.

I. *When* $\rho < 1$. By the definition of a limit (§ 29, p. 19) we can choose $n$ so large, say $n = m$, that when $n \geqq m$ the ratio $\frac{u_{n+1}}{u_n}$ shall differ from $\rho$ by as little as we please, and therefore be less than a proper fraction $r$. Hence

$$u_{m+1} < u_m r; \quad u_{m+2} < u_{m+1} r < u_m r^2; \quad u_{m+3} < u_m r^3;$$

and so on. Therefore, after the term $u_m$, each term of the series $(A)$ is less than the corresponding term of the geometrical series

$$(B) \qquad u_m r + u_m r^2 + u_m r^3 + \cdots.$$

But since $r < 1$, the series $(B)$, and therefore also the series $(A)$, is convergent.*

---

* When examining a series for convergence we are at liberty to disregard any finite number of terms; the rejection of such terms would affect the *value* but not the *existence* of the limit.

**II.** *When* $\rho > 1$ *(or* $\rho = \infty$*).* Following the same line of reasoning as in I, the series $(A)$ may be shown to be divergent.

**III.** *When* $\rho = 1$, the series may be either convergent or divergent; that is, there is no test. For, consider the series $(G)$ on p. 225, namely,

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots+\frac{1}{n^p}+\frac{1}{(n+1)^p}+\cdots.$$

The test ratio is $\frac{u_{n+1}}{u_n}=\frac{(n+1)^p}{n^p}=\left(1+\frac{1}{n}\right)^p$; and

$$\underset{n=\infty}{\text{limit}}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n=\infty}{\text{limit}}\left(1+\frac{1}{n}\right)^p=(1)^p=1\,(=\rho).$$

Hence $\rho = 1$ no matter what value $p$ may have. But on p. 223 we showed that

when $\rho > 1$, the series converges, and
when $\rho \overset{=}{<} 1$, the series diverges.

Thus it appears that $\rho$ can equal unity both for convergent and for divergent series, *and the ratio test for convergence fails.* There are other tests to apply in cases like this, but the scope of our book does not admit of their consideration.

Our results may then be stated in compact form as follows:

*Given the series of positive terms*

$$u_1+u_2+u_3+\cdots+u_n+u_{n+1}+\cdots;$$

*find the limit* $$\underset{n=\infty}{\text{limit}}\left(\frac{u_{n+1}}{u_n}\right)=\rho.$$

I. *When* $\rho < 1$,* *the series is convergent.*
II. *When* $\rho > 1$, *the series is divergent.*
III. *When* $\rho = 1$, *there is no test.*

**Ex. 1.** Test the following series for convergence:

$$e=1+\frac{1}{1}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots+\frac{1}{\underline{|n-1}}+\frac{1}{\underline{|n}}+\cdots.$$

---

* It is not enough that $u_{n+1}/u_n$ becomes and remains less than unity for all values of $n$, but this test requires that the *limit* of $u_{n+1}/u_n$ shall be less than unity. For instance, in the case of the harmonic series this ratio is always less than unity and yet the series diverges as we have seen. The *limit*, however, is not less than unity but equals unity.

*Solution.* The $n$th term is $\dfrac{1}{\underline{|n-1}}$; therefore

$$\underset{n=\infty}{\text{limit}}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n=\infty}{\text{limit}}\left(\frac{\dfrac{1}{\underline{|n}}}{\dfrac{1}{\underline{|n-1}}}\right)=\underset{n=\infty}{\text{limit}}\left(\frac{\underline{|n-1}}{\underline{|n}}\right)=\underset{n=\infty}{\text{limit}}\left(\frac{1}{n}\right)=0\,(=\rho),$$

and by I, p. 225, the series is convergent.

**Ex. 2.** Test the series $\quad \dfrac{\underline{|1}}{10}+\dfrac{\underline{|2}}{10^2}+\dfrac{\underline{|3}}{10^3}+\cdots.$

*Solution.* The $n$th term is here $\dfrac{\underline{|n}}{10^n}$; therefore

$$\underset{n=\infty}{\text{limit}}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n=\infty}{\text{limit}}\left(\frac{\underline{|n+1}}{10^{n+1}}\times\frac{10^n}{\underline{|n}}\right)=\underset{n=\infty}{\text{limit}}\left(\frac{n+1}{10}\right)=\infty,$$

and by II, p. 225, the series is divergent.

**Ex. 3.** Test the series

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots. \tag{C}$$

*Solution.* Here the $n$th term is $\dfrac{1}{(2n-1)2n}$; therefore

$$\underset{n=\infty}{\text{limit}}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n=\infty}{\text{limit}}\left(\frac{2n-1}{2n+1}\right)=\underset{n=\infty}{\text{limit}}\left(1-\frac{2}{2n+1}\right)=1.$$

This gives no test (III, p. 225). But if we compare series $(C)$ with $(G)$, p. 223, making $p=2$, namely,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots, \tag{D}$$

we see that $(C)$ must be convergent since its terms are less than the corresponding terms of $(D)$, which was proven convergent.

**153. Alternating series.** This is the name given to a series whose terms are alternately positive and negative. Such series occur frequently in practice and are of considerable importance.

*If* $\qquad u_1-u_2+u_3-u_4+\cdots$

*is an alternating series whose terms never increase in numerical value,*

*and if* $\qquad \underset{n=\infty}{\text{limit}}\,u_n=0,$

*then the series is convergent.*

*Proof.* The sum of $2n$ (an even number) terms may be written in the two forms

$$(A)\quad S_{2n}=(u_1-u_2)+(u_3-u_4)+(u_5-u_6)+\cdots+(u_{2n-1}-u_{2n}),\ \text{or},$$

$$(B)\quad S_{2n}=u_1-(u_2-u_3)-(u_4-u_5)-\cdots-u_{2n}.$$

Since each difference is positive (if it is not zero, and the assumption $\lim_{n=\infty} u_n = 0$ excludes equality of the terms of the series), series $(A)$ shows that $S_{2n}$ is positive and increases with $n$, while series $(B)$ shows that $S_{2n}$ is always less than $u_1$; therefore by Theorem I, p. 221, $S_{2n}$ must approach a limit less than $u_1$ when $n$ increases, and the series is convergent.

**Ex. 1.** Test the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

*Solution.* Since each term is less in numerical value than the preceding one, and

$$\lim_{n=\infty}(u_n) = \lim_{n=\infty}\left(\frac{1}{n}\right) = 0,$$

the series is convergent.

**154. Absolute convergence.** A series is said to be *absolutely*[*] or *unconditionally* convergent when the series formed from it by making all its terms positive is convergent. Other convergent series are said to be *not absolutely convergent* or *conditionally convergent.* To this latter class belong some convergent alternating series. For example, the series

$$1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots$$

is *absolutely convergent* since the series $(C)$, p. 223, namely,

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots$$

is convergent. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is *conditionally* convergent since the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is divergent (p. 222).

*A series with terms of different signs is convergent if the series deduced from it by making all the signs positive is convergent.*

The proofs of this and the following theorem are omitted.

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[*] The terms of the new series are the numerical (absolute) values of the terms of the given series.

Without placing any restriction on the signs of the terms of the series, the tests given on p. 225 may be stated in the following more general form:

*Given the series*

$$u_1 + u_2 + u_3 + u_4 + \cdots + u_n + u_{n+1} + \cdots;$$

*calculate the limit* $$\underset{n=\infty}{\text{limit}}\left(\frac{u_{n+1}}{u_n}\right) = \rho.$$

I. *When* $|\rho| < 1$, *the series is absolutely convergent.*

II. *When* $|\rho| > 1$, *the series is divergent.*

III. *When* $|\rho| = 1$, *there is no test.*

**155. Power series.** A series of ascending integral powers of a variable, say $x$, of the form

$$(A) \qquad a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where the coefficients $a_0, a_1, a_2, \cdots$ are independent of $x$, is called a *power series in x*. Such series are of prime importance in the further study of the Calculus.

In special cases a power series in $x$ may converge for all values of $x$, but in general it will converge for some values of $x$ and be divergent for other values of $x$. We shall examine $(A)$ only for the case when the coefficients are such that

$$\underset{n=\infty}{\text{limit}}\left(\frac{a_{n+1}}{a_n}\right) = L,$$

where $L$ is a definite number. In $(A)$

$$\underset{n=\infty}{\text{limit}}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n=\infty}{\text{limit}}\left(\frac{a_{n+1}x^{n+1}}{a_nx^n}\right) = \underset{n=\infty}{\text{limit}}\left(\frac{a_{n+1}}{a_n}\right)\cdot x = Lx.$$

Referring to tests I, II, III, we have in this case

$$\rho = Lx,$$

and hence the series $(A)$ is

I. *Absolutely convergent when* $|Lx| < 1$, *or* $|x| < \left|\frac{1}{L}\right|$; *i.e. when* $x$ *lies between* $-\left|\frac{1}{L}\right|$ *and* $+\left|\frac{1}{L}\right|$.

II. *Divergent when* $|Lx| > 1$, *or* $|x| > \left|\frac{1}{L}\right|$; *i.e. when* $x$ *is less than* $-\left|\frac{1}{L}\right|$ *or greater than* $+\left|\frac{1}{L}\right|$.

**III.** *No test when* $|Lx| = 1$, *or* $|x| = \left|\frac{1}{L}\right|$; *i.e. when* $x = \pm\left|\frac{1}{L}\right|$.

**Note.** When $L = 0$, it is evident from I (p. 228) that the power series is absolutely convergent for all finite values of $x$.

**Ex. 1.** Test the series

$$(B) \qquad x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots.$$

*Solution.* The series formed by the coefficients is

$$(C) \qquad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots. \quad \text{Here}$$

$$\operatorname*{limit}_{n=\infty}\left(\frac{a_{n+1}}{a_n}\right) = \operatorname*{limit}_{n=\infty}\left[-\frac{n^2}{(n+1)^2}\right] = \operatorname*{limit}_{n=\infty}\left[-\left(1-\frac{1}{n+1}\right)^2\right] = -1\,(=L).$$

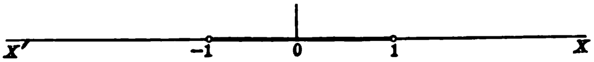
$$\therefore \left|\frac{1}{L}\right| = \left|\frac{1}{-1}\right| = 1.$$

By I the series is absolutely convergent when $x$ lies between $-1$ and $+1$.

By II the series is divergent when $x$ is less than $-1$ or greater than $+1$.

By III there is no test when $x = \pm 1$. But in either case $(B)$ is convergent from the first theorem under § 154, p. 227, since $(D)$, p. 226, was proved convergent.

The series in the above example is said to have $[-1, 1]$ as the *interval of convergence*. This may be written $-1 \leq x \leq 1$, or indicated graphically as follows:



### EXAMPLES

Show that the following nine series are convergent.

1. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$

2. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots.$

3. $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots.$

4. $\frac{1}{3} + \frac{1\cdot 3}{3\cdot 6} + \frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9} + \cdots.$

5. $\frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|7}} + \cdots.$

6. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots.$

7. $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots.$

8. $\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} - \frac{1}{4}\cdot\frac{1}{2^4} + \cdots.$

9. $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \cdots.$

Show that the following three series are divergent.

10. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots.$

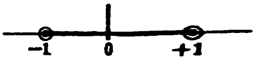
11. $\frac{\underline{|2}}{10} + \frac{\underline{|3}}{10^2} + \frac{\underline{|4}}{10^3} + \cdots.$

12. $1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \frac{1+4}{1+4^2} + \cdots.$

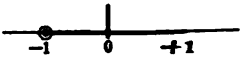
For what values of the variable are the following series convergent?

Graphical representations of intervals of convergence.*

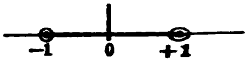
13. $1 + x + x^2 + x^3 \cdots.$ *Ans.* $-1 < x < 1.$



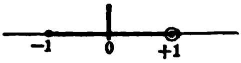
14. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$ *Ans.* $-1 < x \leq 1.$



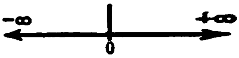
15. $x + x^4 + x^9 + x^{16} + \cdots.$ *Ans.* $-1 < x < 1.$



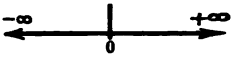
16. $x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \cdots.$ *Ans.* $-1 \leq x < 1.$



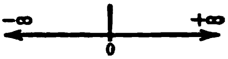
17. $1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots.$ *Ans.* All values of $x$.



18. $1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \cdots.$ *Ans.* All values of $\theta$.

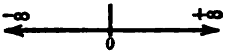


19. $\phi - \frac{\phi^3}{\underline{|3}} + \frac{\phi^5}{\underline{|5}} - \frac{\phi^7}{\underline{|7}} + \cdots.$ *Ans.* All values of $\phi$.

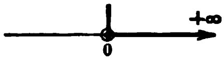


20. $\frac{\sin \alpha}{1^2} - \frac{\sin 3\alpha}{3^2} + \frac{\sin 5\alpha}{5^2} - \cdots.$

*Ans.* All values of $\alpha$.



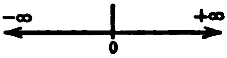
21. $\frac{\cos x}{e^x} + \frac{\cos 2x}{e^{2x}} + \frac{\cos 3x}{e^{3x}} + \cdots.$ *Ans.* $x > 0.$



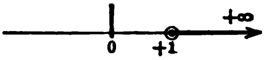
*Hint.* Neither the sine nor cosine can exceed 1 numerically.

22. $1 + x \log a + \frac{x^2 \log^2 a}{\underline{|2}} + \frac{x^3 \log^3 a}{\underline{|3}} + \cdots.$

*Ans.* All values of $x$.

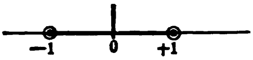


23. $\frac{1}{1 + x} + \frac{1}{1 + x^2} + \frac{1}{1 + x^3} + \cdots.$ *Ans.* $x > 1.$



24. $x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \cdots.$

*Ans.* $-1 < x < 1.$



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* End points that are not included in the interval of convergence have circles drawn about them.

# CHAPTER XX

## EXPANSION OF FUNCTIONS

**156. Introduction.** The student is already familiar with some methods of expanding certain functions into series. Thus, by the Binomial Theorem,

$$(A) \qquad (a+x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4,$$

giving a finite power series from which the exact value of $(a+x)^4$ for any value of $x$ may be calculated. Also by actual division,

$$(B) \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + \left(\frac{1}{1-x}\right)x^n,$$

we get an equivalent series, all of whose coefficients except that of $x^n$ are constants, $n$ being a positive integer.

Suppose we wish to calculate the value of this function when $x = .5$, not by substituting directly in

$$\frac{1}{1-x},$$

but by substituting $x = .5$ in the equivalent series

$$(C) \qquad (1 + x + x^2 + x^3 + \cdots + x^{n-1}) + \left(\frac{1}{1-x}\right)x^n.$$

Assuming $n = 8$, $(C)$ gives for $x = .5$

$$(D) \qquad \frac{1}{1-x} = 1.9921875 + .0078125.$$

If we then assume the value of the function to be the sum of the first eight terms of series $(C)$, the error we make is .0078125. However, in case we need the value of the function correct to two decimal places only, the number 1.99 is as close an approximation to the true value as we care for since the error is less than .01. It is evident that if a greater degree of accuracy is desired, all we need to do is to use more terms of the power series

$$(E) \qquad 1 + x + x^2 + x^3 + \cdots.$$

Since, however, we see at once that

$$\left[\frac{1}{1-x}\right]_{x=.5}=2,$$

there is no necessity for the above discussion, except for purposes of illustration. As a matter of fact the process of computing the value of a function from an equivalent series into which it has been expanded is of the greatest practical importance, the values of the elementary transcendental functions such as the sine, cosine, logarithm, etc., being computed most simply in this way.

So far we have learned how to expand only a few special forms into series; we shall now consider a method of expansion applicable to an extensive and important class of functions and called *Taylor's Theorem.*

**157. Taylor's Theorem* and Taylor's Series.** Replacing $b$ by $x$ in $(E)$, p. 169, the *extended theorem of the mean* takes on the form

$$(59)\quad f(x)=f(a)+\frac{(x-a)}{\underline{|1}}f'(a)+\frac{(x-a)^2}{\underline{|2}}f''(a)+\frac{(x-a)^3}{\underline{|3}}f'''(a)+\cdots$$
$$+\frac{(x-a)^{n-1}}{\underline{|n-1}}f^{(n-1)}(a)+\frac{(x-a)^n}{\underline{|n}}f^{(n)}(x_1),$$

where $x_1$ lies between $a$ and $x$. (59), which is one of the most far-reaching theorems in the Calculus, is called *Taylor's Theorem.* We see that it expresses $f(x)$ as the sum of a finite series in $(x-a)$.

The last term in (59), namely, $\frac{(x-a)^n}{\underline{|n}}f^{(n)}(x_1)$, is sometimes called the *remainder in Taylor's Theorem after n terms.* If this remainder converges towards zero as the number of terms increases without limit, then the right-hand side of (59) becomes an infinite power series called *Taylor's Series.*† In that case we may write (59) in the form

$$(60)\quad f(x)=f(a)+\frac{(x-a)}{\underline{|1}}f'(a)+\frac{(x-a)^2}{\underline{|2}}f''(a)+\frac{(x-a)^3}{\underline{|3}}f'''(a)+\cdots,$$

and we say that *the function has been expanded into a Taylor's Series.* For all values of $x$ for which the remainder approaches zero as $n$ increases without limit, this series converges and its sum gives the exact value of $f(x)$, because the difference (= the remainder) between the function and the sum of $n$ terms of the series approaches the limit zero (§ 30, p. 21).

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* Also known as *Taylor's Formula.*

† Published by Dr. Brook Taylor (1685–1731) in his *Methodus Incrementorum*, London, 1715.

On the other hand, if the series converges for values of $x$ for which the remainder does not approach zero as $n$ increases without limit, then the limit of the sum of the series is not equal to the function $f(x)$.

*The infinite series* (60) *represents the function for those values of $x$ and those only for which the remainder approaches zero as the number of terms increases without limit.*

It is usually easier to determine the interval of convergence of the series than that for which the remainder approaches zero; *but in simple cases the two intervals are identical* (see footnote, p. 236).

When the values of a function and its successive derivatives are known for some value of the variable, as $x = a$, then (60) is used for finding the value of the function for values of $x$ near $a$, and (60) is also called *the expansion of $f(x)$ in the vicinity of $x = a$.*

Ex. 1. Expand $\log x$ in powers of $(x - 1)$.

*Solution.*

$$\begin{aligned} f(x) &= \log x, & f(1) &= 0; \\ f'(x) &= \frac{1}{x}, & f'(1) &= 1; \\ f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1; \\ f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2. \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

Substituting in (60), $\log x = x - 1 - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3 - \cdots$. *Ans.*

This converges for values of $x$ between 0 and 2 (§ 155, p. 228) and is the *expansion of $\log x$ in the vicinity of $x = 1$*, the remainder converging to zero.

When a function of the sum of two numbers $a$ and $x$ is given, say $f(a + x)$, it is frequently desirable to expand the function into a power series in one of them, say $x$. For this purpose we use another form of Taylor's Series, gotten by replacing $x$ by $a + x$ in (60), namely,

$$(61)\qquad f(a + x) = f(a) + \frac{x}{\underline{|1}}f'(a) + \frac{x^2}{\underline{|2}}f''(x) + \frac{x^3}{\underline{|3}}f'''(a) + \cdots.$$

**Ex. 1.** Expand $\sin(a+x)$ in powers of $x$.

*Solution.* Here $f(a+x)=\sin(a+x)$;
hence, placing $x=0$,

$$f(a)=\sin a,$$
$$f'(a)=\cos a,$$
$$f''(a)=-\sin a,$$
$$f'''(a)=-\cos a,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Substituting in (61),

$$\sin(a+x)=\sin a+\frac{x}{1}\cos a-\frac{x^2}{\underline{|2}}\sin a-\frac{x^3}{\underline{|3}}\cos a+\cdots. \quad \textit{Ans.}$$

## EXAMPLES*

1. Expand $e^x$ in powers of $x-2$. *Ans.* $e^x=e^2+e^2(x-2)+\frac{e^2}{\underline{|2}}(x-2)^2+\cdots$.

2. Expand $x^3-2x^2+5x-7$ in powers of $x-1$.
*Ans.* $-3+4(x-1)+(x-1)^2+(x-1)^3$.

3. Expand $3y^2-14y+7$ in powers of $y-3$.
*Ans.* $-8+4(y-3)+3(y-3)^2$.

4. Expand $5z^2+7z+z$ in powers of $z-2$.
*Ans.* $37+27(z-2)+5(z-2)^2$.

5. Expand $\cos(a+x)$ in powers of $x$.
*Ans.* $\cos(a+x)=\cos a-x\sin a-\frac{x^2}{\underline{|2}}\cos a+\frac{x^3}{\underline{|3}}\sin a+\cdots$.

6. Expand $\log(x+h)$ in powers of $x$.
*Ans.* $\log(x+h)=\log h+\frac{x}{h}-\frac{x^2}{2h^2}+\frac{x^3}{3h^3}+\cdots$.

7. Expand $\tan(x+h)$ in powers of $h$.
*Ans.* $\tan(x+h)=\tan x+h\sec^2 x+h^2\sec^2 x\tan x+\cdots$.

8. Expand the following in powers of $h$.

(a) $(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{\underline{|2}}x^{n-2}h^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}h^3+\cdots$.

(b) $e^{x+h}=e^x\left(1+h+\frac{h^2}{\underline{|2}}+\frac{h^3}{\underline{|3}}+\cdots\right)$.

**158. Maclaurin's Theorem and Maclaurin's Series.** A particular case of Taylor's Theorem is found by placing $a=0$ in (59), p. 232, giving

$$(62)\qquad f(x)=f(0)+\frac{x}{\underline{|1}}f'(0)+\frac{x^2}{\underline{|2}}f''(0)+\frac{x^3}{\underline{|3}}f'''(0)+\cdots$$
$$+\frac{x^{n-1}}{\underline{|n-1}}f^{(n-1)}(0)+\frac{x^n}{\underline{|n}}f^{(n)}(x_1),$$

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* In these examples we assume that the functions can be developed into a power series.

where $x_1$ lies between 0 and $x$. (62) is called *Maclaurin's Theorem.* The right-hand member is evidently a series in $x$ in the same sense that (59), p. 232, is a series in $x - a$.

Placing $a = 0$ in (60), p. 232, we get *Maclaurin's Series,**

$$(63) \qquad f(x) = f(0) + \frac{x}{\lfloor\!\underline{1}} f'(0) + \frac{x^2}{\lfloor\!\underline{2}} f''(0) + \frac{x^3}{\lfloor\!\underline{3}} f'''(0) + \cdots,$$

a special case of Taylor's Series that is very useful. The statements made concerning the remainder and the convergence of Taylor's Series apply with equal force to Maclaurin's Series, the latter being merely a special case of the former.

The student should not fail to note the importance of such an expansion as (63). In all practical computations results correct to a certain number of decimal places are sought, and since the process in question replaces a function perhaps difficult to calculate by *an ordinary polynomial with constant coefficients*, it is very useful in simplifying such computations. Of course we must use terms enough to give the desired degree of accuracy.

In the case of an alternating series (§ 153, p. 226) the error made by stopping at any term is numerically less than that term, since the sum of the series after that term is numerically less than that term.

Ex. 1. **Expand $\cos x$ into an infinite power series and determine for what values of $x$ it converges.**

*Solution.* Differentiating first and then placing $x = 0$, we get

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1,\\ f'(x) &= -\sin x, & f'(0) &= 0,\\ f''(x) &= -\cos x, & f''(0) &= -1,\\ f'''(x) &= \sin x, & f'''(0) &= 0,\\ f^{\text{iv}}(x) &= \cos x, & f^{\text{iv}}(0) &= 1,\\ f^{\text{v}}(x) &= -\sin x, & f^{\text{v}}(0) &= 0,\\ f^{\text{vi}}(x) &= -\cos x, & f^{\text{vi}}(0) &= -1,\\ &\text{etc.}, & &\text{etc.} \end{aligned}$$

Substituting in (63),

$$(A) \qquad \cos x = 1 - \frac{x^2}{\lfloor\!\underline{2}} + \frac{x^4}{\lfloor\!\underline{4}} - \frac{x^6}{\lfloor\!\underline{6}} + \cdots.$$

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* Named after Colin Maclaurin (1698–1746), being first published in his *Treatise of Fluxions*, Edinburgh, 1742. The Series is really due to Stirling (1692–1770).

Comparing with Ex. 18, p. 230, we see that the series converges for all values of $x$.

In the same way for $\sin x$.

$$(B)\qquad \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots,$$

which converges for all values of $x$ (Ex. 19, p. 230).[*]

Ex. 2. Using the series ($B$) found in the last example, calculate sin 1 correct to four decimal places.

*Solution.* Here $x = 1$; that is, the angle is expressed in circular measure (see second footnote, p. 17). Therefore, substituting $x = 1$ in ($B$) of the last example,

$$\sin 1 = 1 - \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} - \frac{1}{\underline{|7}} + \cdots.$$

Summing up the positive and negative terms separately,

$$\begin{array}{rr} 1 = 1.00000\cdots & \dfrac{1}{\underline{|3}} = 0.16667\cdots \\[2ex] \dfrac{1}{\underline{|5}} = 0.00833\cdots & \dfrac{1}{\underline{|7}} = 0.00019\cdots \\ \hline 1.00833\cdots & 0.16686\cdots \end{array}$$

Hence $$\sin 1 = 1.00833 - 0.16686 = 0.84147\cdots,$$

which is correct to four decimal places since the error made must be less than $\frac{1}{\underline{|9}}$, i.e. less than .000003. Obviously the value of sin 1 may be calculated to any desired degree of accuracy by simply including a sufficient number of additional terms.

## EXAMPLES

Verify the following expansions of functions into power series by Maclaurin's Series and determine for what values of the variable they are convergent.

1. $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots.$ Convergent for all values of $x$.

2. $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} - \cdots.$ Convergent for all values of $x$.

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[*] Since here $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$ and $f^{(n)}(x_1) = \sin\left(x_1 + \frac{n\pi}{2}\right)$, we have, by substituting in the last term of (62), p. 234,

$$\text{remainder} = \frac{x^n}{\underline{|n}} \sin\left(x_1 + \frac{n\pi}{2}\right). \qquad 0 < x_1 < x$$

But $\sin\left(x_1 + \frac{n\pi}{2}\right)$ can never exceed unity, and from Ex. 17, p. 230, $\underset{n=\infty}{\text{limit}} \frac{x^n}{\underline{|n}} = 0$ for all values of $x$. Hence

$$\underset{n=\infty}{\text{limit}} \frac{x^n}{\underline{|n}} \sin\left(x_1 + \frac{n\pi}{2}\right) = 0$$

for all values of $x$; that is, in this case *the limit of the remainder is* 0 *for all values of $x$ for which the series converges.* This is also the case for all the functions considered in this book.

3. $a^x = 1 + x \log a + \frac{x^2 \log^2 a}{\underline{|2}} + \frac{x^3 \log^3 a}{\underline{|3}} + \cdots.$ Convergent for all values of $x$.

4. $\sin kx = kx - \frac{k^3x^3}{\underline{|3}} + \frac{k^5x^5}{\underline{|5}} - \frac{k^7x^7}{\underline{|7}} + \cdots.$ Convergent for all values of $x$, $k$ being any constant.

5. $e^{-kx} = 1 - kx + \frac{k^2x^2}{\underline{|2}} - \frac{k^3x^3}{\underline{|3}} + \frac{k^4x^4}{\underline{|4}} - \cdots.$ Convergent for all values of $x$, $k$ being any constant.

6. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$ Convergent if $-1 < x \leqq 1$.

7. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots.$ Convergent if $-1 \leqq x < 1$.

8. $\text{arc} \sin x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \cdots.$ Convergent if $-1 < x < 1$.

9. $\text{arc} \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots.$ Convergent if $-1 \leqq x \leqq 1$.

10. $\sin^2 x = x^2 - \frac{2 x^4}{\underline{|3}} + \frac{32 x^6}{\underline{|6}} + \cdots.$ Convergent for all values of $x$.

11. $e^{\sin \phi} = 1 + \phi + \frac{\phi^2}{2} - \frac{\phi^4}{8} + \cdots.$ Convergent for all values of $\phi$.

12. $e^\theta \sin \theta = \theta + \theta^2 + \frac{\theta^3}{3} - \frac{4 \theta^5}{\underline{|5}} - \frac{8 \theta^6}{\underline{|6}} - \cdots.$ Convergent for all values of $\theta$.

13. Show that $\log x$ cannot be expanded by Maclaurin's Theorem.

Compute the values of the following functions by substituting directly in the equivalent power series, taking terms enough until the results agree with those given below.

14. $e = 2.7182 \cdots.$

*Solution.* Let $x = 1$ in series of Ex. 1; then

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \cdots.$$

First term $= 1.00000$
Second term $= 1.00000$
Third term $= 0.50000$
Fourth term $= 0.16667 \cdots$ [Dividing third term by 3.]
Fifth term $= 0.04167 \cdots$ [Dividing fourth term by 4.]
Sixth term $= 0.00833 \cdots$ [Dividing fifth term by 5.]
Seventh term $= 0.00139 \cdots$ [Dividing sixth term by 6.]
Eighth term $= 0.00019 \cdots$, etc. [Dividing seventh term by 7.]
Adding, $e = 2.71825 \cdots$ *Ans.*

15. $\text{arc} \tan (\frac{1}{5}) = 0.1973 \cdots$; use series in Ex. 9.

16. $\cos 1 = 0.5403 \cdots$; use series in Ex. 2.

17. $\cos 10° = 0.9848 \cdots$; use series in Ex. 2.

**18.** $\sin\frac{\pi}{4} = 0.6946\cdots$; use series $(B)$, p. 236.

**19.** $\sin .5 = 0.4794\cdots$; use series $(B)$, p. 236.

**20.** $e^2 = 1 + 2 + \frac{2^2}{\lfloor\underline{2}} + \frac{2^3}{\lfloor\underline{3}} + \cdots = 7.3891.$

**21.** $\sqrt{e} = 1 + \frac{1}{2} + \frac{1}{2^2\lfloor\underline{2}} + \frac{1}{2^3\lfloor\underline{3}} + \cdots = 1.6487.$

**159. Computation by series.**

I. *The computation of π by series.*

From Ex. 8, p. 237, we have

$$\text{arc}\sin x = x + \frac{1\cdot x^3}{2\cdot 3} + \frac{1\cdot 3\, x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5\, x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots.$$

Since this series converges * for values of $x$ between $-1$ and $+1$, we may let $x = \frac{1}{2}$, giving

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\left(\frac{1}{2}\right)^5 + \cdots,$$

or, $\pi = 3.1415\ \cdots.$

Evidently we might have used the series of Ex. 9, p. 237, instead. Both of these series converge rather slowly, but there are other series, found by more elaborate methods, by means of which the correct value of $\pi$ to a large number of decimal places may be easily calculated.

II. *The computation of logarithms by series.*

Series play a very important rôle in making the necessary calculations for the construction of logarithmic tables.

From Ex. 6, p. 237, we have

$$(A) \qquad \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

This series converges for $x = 1$, and we can find log 2 by placing $x = 1$ in $(A)$, giving

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots.$$

But this series is not well adapted to numerical computation, because it converges so slowly that it would be necessary to take

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* We assume that it converges to the correct value.

1000 terms in order to get the value of log 2 correct to three decimal places. A rapidly converging series for computing logarithms will now be deduced.

By the theory of logarithms,

$$(B)\qquad \log\frac{1+x}{1-x}=\log(1+x)-\log(1-x). \qquad 8,\text{ p. } 2$$

Substituting in ($B$) the equivalent series for $\log(1+x)$ and $\log(1-x)$ found in Exs. 6 and 7 on p. 237, we get*

$$(C)\qquad \log\frac{1+x}{1-x}=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots\right],$$

which is convergent when $x$ is numerically less than unity. Let

$$(D)\qquad \frac{1+x}{1-x}=\frac{M}{N},\text{ whence } x=\frac{M-N}{M+N},$$

and we see that $x$ will always be numerically less than unity for all positive values of $M$ and $N$. Substituting from ($D$) into ($C$), we get

$$(E)\quad \log\frac{M}{N}=\log M-\log N$$
$$=2\left[\frac{M-N}{M+N}+\frac{1}{3}\left(\frac{M-N}{M+N}\right)^3+\frac{1}{5}\left(\frac{M-N}{M+N}\right)^5+\cdots\right],$$

a series which is convergent for all positive values of $M$ and $N$; and it is always possible to choose $M$ and $N$ so as to make it converge rapidly.

Placing $M=2$ and $N=1$ in ($E$), we get

$$\log 2=2\left[\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\frac{1}{7}\cdot\frac{1}{3^7}+\cdots\right]=0.69314718\ldots.$$

$$\left[\text{Since } \log N=\log 1=0,\text{ and } \frac{M-N}{M+N}=\frac{1}{3}.\right]$$

Placing $M=3$ and $N=2$ in ($E$), we get

$$\log 3=\log 2+2\left[\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}+\cdots\right]=1.09861229\ldots.$$

---

* The student should notice that we have treated the series as if they were ordinary sums, but they are not; they are *limits* of sums. To justify this step is beyond the scope of this book.

It is only necessary to compute the logarithms of prime numbers in this way, the logarithms of composite numbers being then found by using theorems 7–10, p. 2. Thus,

$$\log 8 = \log 2^3 = 3 \log 2 = 2.07944154 \cdots,$$
$$\log 6 = \log 3 + \log 2 = 1.79175947 \cdots.$$

All the above are *Naperian* or *natural logarithms*, i.e. the base is $e = 2.7182818$. If we wish to find *Briggs'* or *common logarithms*, where the base 10 is employed, all we need to do is to change the base by means of the formula

$$\log_{10} n = \frac{\log_e n}{\log_e 10}.$$

Thus, $$\log_{10} 2 = \frac{\log_e 2}{\log_e 10} = \frac{0.693 \cdots}{2.302 \cdots} = 0.301 \cdots.$$

In the actual computation of a table of logarithms only a few of the tabulated values are calculated from series, all the rest being found by employing theorems in the theory of logarithms and various ingenious devices designed for the purpose of saving work.

### EXAMPLES

Calculate by the methods of this article the following logarithms.

1. $\log_e 5 = 1.6094 \cdots.$
2. $\log_e 10 = 2.3025 \cdots.$
3. $\log_e 24 = 3.1781 \cdots.$
4. $\log_{10} 5 = 0.6990 \cdots.$

**160. Approximate formulas derived from series.** In the two preceding sections we evaluated a function from its equivalent power series by substituting the given value of $x$ in a certain number of the first terms of that series, the number of terms taken depending on the degree of accuracy required. It is of great practical importance to note that this really means that *we are considering the function as approximately equal to an ordinary polynomial with constant coefficients.* For example, consider the series

$$(A) \qquad \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots.$$

This is an alternating series for both positive and negative values of $x$. Hence the error made if we assume $\sin x$ to be approximately equal to the sum of the first $n$ terms is numerically less than the $(n+1)$th term, § 153, p. 227. For example, assume

$$(B)\qquad \sin x = x,$$

and let us find for what values of $x$ this is correct to three places of decimals. To do this, set

$$(C)\qquad \left|\frac{x^3}{\underline{|3}}\right| < .001.$$

This gives $x$ numerically less than $\sqrt[3]{.006}\,(= .1817)$; that is, $(B)$ is correct to three decimal places when $x$ lies between $+10°.4$ and $-10°.4$.

The error made in neglecting all terms in $(A)$ after the one in $x^{n-1}$ is given by the remainder, (62), p. 234,

$$(D)\qquad R = \frac{x^n}{\underline{|n}} f^{(n)}(x_1);$$

hence we can find for what values of $x$ a polynomial represents the function to any desired degree of accuracy by writing the inequality

$$(E)\qquad |R| < \textit{limit of error},$$

and solving for $x$, provided we know the maximum value of $f^{(n)}(x_1)$. Thus, if we wish to find for what values of $x$ the formula

$$(F)\qquad \sin x = x - \frac{x^3}{6}$$

is correct to two decimal places (i.e. error $< .01$), knowing that $|f^{(v)}(x_1)| \leq 1$, we have from $(D)$ and $(E)$,

$$\frac{|x^5|}{120} < .01;\ \text{that is, } |x| < \sqrt[5]{1.2};\ \text{or, } |x| \leq 1.$$

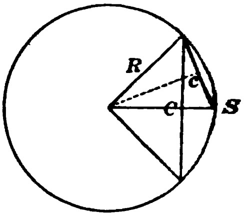
Therefore $x - \frac{x^3}{6}$ gives the correct value of $\sin x$ to two decimal places if $|x| \leq 1$, i.e. if $x$ lies between $+57°$ and $-57°$. This agrees with the discussion of $(A)$ as an alternating series.

Since in a great many practical problems accuracy to two or three decimal places only is required, the usefulness of such approximate formulas as $(B)$ and $(F)$ is apparent.

*Huygens' approximation to the length of a circular arc.*

Let $S$ denote the length of the required arc, $C$ its chord, $c$ the chord of half the arc, and $R$ the radius of the circle. The circular measure of the whole angle being

$$\frac{S}{R},$$



we have

$$\sin\frac{S}{2R}=\frac{\frac{C}{2}}{R},\quad \sin\frac{S}{4R}=\frac{\frac{c}{2}}{R};\ \text{or,}$$

$$\frac{C}{2R}=\sin\frac{S}{2R},\quad \frac{c}{2R}=\sin\frac{S}{4R}.$$

Developing the right-hand members of the last two equations into power series in $S$ by (68), p. 235, we get

$$(G)\qquad \frac{C}{2R}=\frac{S}{2R}-\frac{S^3}{1\cdot 2\cdot 3\cdot 2^3R^3}+\frac{S^5}{1\cdot 2\cdots 5\cdot 2^5R^5}+\cdots,$$

$$(H)\qquad \frac{c}{2R}=\frac{S}{4R}-\frac{S^3}{1\cdot 2\cdot 3\cdot 2^6R^3}+\frac{S^5}{1\cdot 2\cdots 5\cdot 2^{10}R^5}+\cdots.$$

Multiplying ($H$) by 8 and then subtracting ($H$) from ($G$) in order to eliminate the term containing $S^3$, we have approximately

$$\frac{8c-C}{2R}=\frac{3S}{2R}-\frac{3}{4}\cdot\frac{S^5}{1\cdot 2\cdots 5\cdot 2^5R^5},$$

or,

$$\frac{8c-C}{3}=S-\frac{S^5}{7680R^4}.$$

Hence, for an arc equal in length to the radius, the error in taking

$$(I)\qquad S=\frac{8c-C}{3}$$

is less than $\frac{1}{7680}$ of the whole arc. For an arc of half the length of the radius the proportionate error is one sixteenth less, and so on.*

In practice Huygens' approximation is generally used in the form

$$(J)\qquad S=2c+\tfrac{1}{3}(2c-C).$$

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* For an angle of 30° the error is less than 1 in 100,000, for 45° less than 1 in 20,000, and for 60 less than 1 in 6000

This simple method of finding approximately the length of an arc of a circle is much employed. To find the approximate length of a portion of any continuous curve, divide it into an even number of suitable arcs, regarding the arcs as approximately circular.

**Ex. 1.** Find by Huygens' approximation the length of an arc of 30° in a circle whose radius is 100,000 ft.

*Solution.* Here $c = 2R \sin 7° 30'$, $C = 2R \sin 15°$; but from tables of natural sines we get $\sin 7° 30' = .1305268$, $\sin 15° = .2588190$.

Substituting in $(J)$, $s = 52359.71$. The true value, assuming $\pi = 3.1415926$, is $52359.88$; hence the error is but .17 ft., or about 2 inches.

## EXAMPLES

1. Draw the graphs of the functions $x$, $x - \frac{x^3}{\underline{|3}}$, $x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}}$ respectively, and compare them with the graph of $\sin x$.

2. If $d$ is the distance between the middle points of the chord $c$ and the circular arc $s$, show that the error in taking

$$c = s - \frac{8}{3}\frac{d^2}{s}$$

is less than $\frac{32}{3}\frac{d^4}{s^3}$.

**161. Taylor's Theorem for functions of two or more variables.** The scope of this book will allow only an elementary treatment of the expansion of functions involving more than one variable by Taylor's Theorem. The expressions for the remainder are complicated and will not be written down.

Having given the function

$$(A) \qquad f(x, y),$$

it is required to expand the function

$$(B) \qquad f(x + h, y + k)$$

in powers of $h$ and $k$.

Consider the function

$$(C) \qquad f(x + ht, y + kt).$$

Evidently $(B)$ is the value of $(C)$ when $t = 1$. Considering $(C)$ as a function of $t$, we may write

$$(D) \qquad f(x + ht, y + kt) = F(t),$$

which may then be expanded in powers of $t$ by Maclaurin's Theorem, (62), p. 234, giving

$$F(t)=F(0)+tF'(0)+\frac{t^2}{\underline{|2}}F''(0)+\frac{t^3}{\underline{|3}}F'''(0)+\cdots. \tag{E}$$

Let us now express the successive derivatives of $F(t)$ with respect to $t$ in terms of the partial derivatives of $F(t)$ with respect to $x$ and $y$. Let

$$\alpha=x+ht,\quad \beta=y+kt; \tag{F}$$

then by (49), p. 199,

$$F'(t)=\frac{\partial F}{\partial \alpha}\frac{d\alpha}{dt}+\frac{\partial F}{\partial \beta}\frac{d\beta}{dt}. \tag{G}$$

But from $(F)$,

$$\frac{d\alpha}{dt}=h \text{ and } \frac{d\beta}{dt}=k; \tag{H}$$

and since $F(t)$ is a function of $x$ and $y$ through $\alpha$ and $\beta$,

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial x} \text{ and } \frac{\partial F}{\partial y}=\frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial y};$$

or, since from $(F)$, $\frac{\partial \alpha}{\partial x}=1$ and $\frac{\partial \beta}{\partial y}=1$,

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial \alpha} \text{ and } \frac{\partial F}{\partial y}=\frac{\partial F}{\partial \beta}. \tag{I}$$

Substituting in $(G)$ from $(I)$ and $(H)$,

$$F'(t)=h\frac{\partial F}{\partial x}+k\frac{\partial F}{\partial y}. \tag{J}$$

Replacing $F(t)$ by $F'(t)$ in $(J)$, we get

$$F''(t)=h\frac{\partial F'}{\partial x}+k\frac{\partial F'}{\partial y}=h\left\{h\frac{\partial^2 F}{\partial x^2}+k\frac{\partial^2 F}{\partial x\partial y}\right\}+k\left\{h\frac{\partial^2 F}{\partial x\partial y}+k\frac{\partial^2 F}{\partial y^2}\right\}.$$

$$\therefore\ F''(t)=h^2\frac{\partial^2 F}{\partial x^2}+2\,hk\frac{\partial^2 F}{\partial x\partial y}+k^2\frac{\partial^2 F}{\partial y^2}. \tag{K}$$

In the same way the third derivative is

$$F'''(t)=h^3\frac{\partial^3 F}{\partial x^3}+3\,h^2k\frac{\partial^3 F}{\partial x^2\partial y}+3\,hk^2\frac{\partial^3 F}{\partial x\partial y^2}+k^3\frac{\partial^3 F}{\partial y^3}, \tag{L}$$

and so on for higher derivatives.

When $t = 0$, we have from $(D)$, $(G)$, $(J)$, $(K)$, $(L)$,

$$F(0) = f(x, y), \text{ i.e. } F(t) \text{ is replaced by } f(x, y),$$

$$F'(0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y},$$

$$F''(0) = h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2},$$

$$F'''(0) = h^3\frac{\partial^3 f}{\partial x^3} + 3\,h^2 k\frac{\partial^3 f}{\partial x^2 \partial y} + 3\,hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3},$$

and so on.

Substituting these results in $(E)$, we get

$$(64)\quad f(x + ht, y + kt) = f(x, y) + t\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{t^2}{\underline{|2}}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots.$$

To get $f(x + h, y + k)$, replace $t$ by 1 in (64), giving *Taylor's Theorem for a function of two independent variables,*

$$(65)\quad f(x + h, y + k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots,$$

which is the required expansion in powers of $h$ and $k$. Evidently (65) is also adapted to the expansion of $f(x + h, y + k)$ in powers of $x$ and $y$ by simply interchanging $x$ with $h$ and $y$ with $k$. Thus,

$$(65\,a)\quad f(x + h, y + k) = f(h, k) + x\frac{\partial f}{\partial h} + y\frac{\partial f}{\partial k} + \frac{1}{\underline{|2}}\left(x^2\frac{\partial^2 f}{\partial h} + 2\,xy\frac{\partial^2 f}{\partial h \partial k} + y^2\frac{\partial^2 f}{\partial k^2}\right) + \cdots.$$

Similarly for three variables we shall find

$$(66)\quad f(x + h, y + k, z + l) = f(x, y, z) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + l^2\frac{\partial^2 f}{\partial z^2} + 2\,hk\frac{\partial^2 f}{\partial x \partial y} + 2\,lh\frac{\partial^2 f}{\partial z \partial x} + 2\,kl\frac{\partial^2 f}{\partial y \partial z}\right) + \cdots,$$

and so on for any number of variables.

**EXAMPLES**

1. Given $f(x, y) \equiv Ax^2 + Bxy + Cy^2$; expand $f(x+h, y+k)$ in powers of $h$ and $k$.

*Solution.*
$$\frac{\partial f}{\partial x} = 2Ax + By, \quad \frac{\partial f}{\partial y} = Bx + 2Cy;$$
$$\frac{\partial^2 f}{\partial x^2} = 2A, \quad \frac{\partial^2 f}{\partial x \partial y} = B, \quad \frac{\partial^2 f}{\partial y^2} = 2C.$$

The third and higher partial derivatives are all zero. Substituting in (65),

$$f(x+h, y+k) \equiv Ax^2 + Bxy + Cy^2 + (2Ax + By)h + (Bx + 2Cy)k$$
$$+ Ah^2 + Bhk + Ck^2. \quad Ans.$$

2. Given $f(x, y, z) \equiv Ax^2 + By^2 + Cz^2$; expand $f(x+l, y+m, z+n)$ in powers of $l, m, n$.

*Solution.*
$$\frac{\partial f}{\partial x} = 2Ax, \quad \frac{\partial f}{\partial y} = 2By, \quad \frac{\partial f}{\partial z} = 2Cz;$$
$$\frac{\partial^2 f}{\partial x^2} = 2A, \quad \frac{\partial^2 f}{\partial y^2} = 2B, \quad \frac{\partial^2 f}{\partial z^2} = 2C, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 0.$$

The third and higher partial derivatives are all zero. Substituting in (66),

$$f(x+l, y+m, z+n) \equiv Ax^2 + By^2 + Cz^2 + 2Axl + 2Bym + 2Czn$$
$$+ Al^2 + Bm^2 + Cn^2. \quad Ans.$$

3. Given $f(x, y) \equiv \sqrt{x} \tan y$; expand $f(x+h, y+k)$ in powers of $h$ and $k$.

4. Given $f(x, y, z) \equiv Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx$; expand $f(x+h, y+k, z+l)$ in powers of $h, k, l$.

**162. Maxima and minima of functions of two independent variables.** The function $f(x, y)$ is said to be a *maximum* at $x = a$, $y = b$ when $f(a, b)$ is greater than $f(x, y)$ for all values of $x$ and $y$ in the neighborhood of $a$ and $b$. Similarly $f(a, b)$ is said to be a *minimum* at $x = a$, $y = b$ when $f(a, b)$ is less than $f(x, y)$ for all values of $x$ and $y$ in the neighborhood of $a$ and $b$.

These definitions may be stated in analytical form as follows:

If, for all values of $h$ and $k$ numerically less than some small positive quantity,

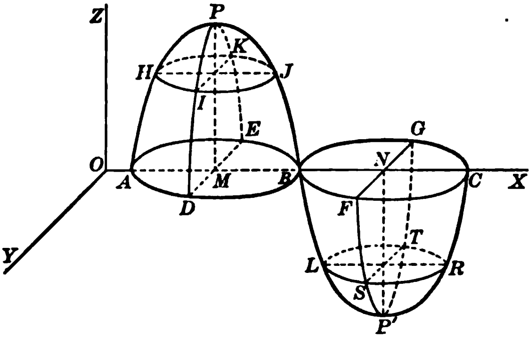
(*A*) $f(a+h, b+k) - f(a, b) =$ *a negative number*, then $f(a, b)$ is a *maximum* value of $f(x, y)$. If

(*B*) $f(a+h, b+k) - f(a, b) =$ *a positive number*, then $f(a, b)$ is a *minimum* value of $f(x, y)$.

These statements may be interpreted geometrically as follows: a point $P$ on the surface

$$z = f(x, y)$$

is a maximum point when it is "higher" than *all* other points on the surface in its neighborhood, the coördinate plane $XOY$ being assumed horizontal. Similarly $P'$ is a minimum point on the surface when it is "lower" than *all* other points on the surface in its neighborhood. It is therefore evident that all vertical planes through $P$ cut the surface in curves (as $APB$ or $DPE$ in the figure), each of which has a maximum ordinate $z$ ($=MP$) at $P$. In the same manner all vertical planes through $P'$ cut the surface in curves (as $BP'C$ or $FP'G$), each of which has a minimum ordinate $z$ ($=NP'$) at $P'$. Also, any contour (as $HIJK$) cut out of the surface by a horizontal plane in the immediate neighborhood of $P$ must be a small closed curve. Similarly we have the contour $LSRT$ near the minimum point $P'$.



It was shown in §§ 93, 94, pp. 117–121, that a ***necessary*** condition that a function of one variable should have a maximum or a minimum for a given value of the variable was that its first derivative should be zero for the given value of the variable. Similarly for a function $f(x, y)$ of two independent variables, a ***necessary*** condition that $f(a, b)$ should be a maximum or a minimum (i.e. a turning value) is that for $x = a$, $y = b$,

$$(C) \qquad \frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0.$$

*Proof.* Evidently $(A)$ and $(B)$ must hold when $k = 0$; that is,

$$f(a + h, b) - f(a, b)$$

is always negative or always positive for all values of $h$ sufficiently small numerically. By §§ 93, 94, a necessary condition for this is that $\frac{d}{dx}f(x, b)$ shall vanish for $x = a$, or, what amounts to the same thing, $\frac{\partial}{\partial x}f(x, y)$ shall vanish for $x = a$, $y = b$. Similarly $(A)$ and $(B)$ must hold when $h = 0$, giving as a second necessary condition that $\frac{\partial}{\partial y}f(x, y)$ shall vanish for $x = a$, $y = b$.

In order to determine *sufficient* conditions that $f(a, b)$ shall be a maximum or a minimum it is necessary to proceed to higher derivatives. To derive sufficient conditions for all cases is beyond the scope of this book.* The following discussion, however, will suffice for all the problems given here.

Expanding $f(a + h, b + k)$ by Taylor's Theorem, (65), p. 245, replacing $x$ by $a$ and $y$ by $b$, we get

$$(D) \quad f(a + h, b + k) = f(a, b) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + R,$$

where the partial derivatives are evaluated for $x = a$, $y = b$, and $R$ denotes the sum of all the terms not written down. All such terms are of a degree higher than the second in $h$ and $k$.

Since $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, from $(C)$, p. 247, we get, after transposing $f(a, b)$,

$$(E) \quad f(a + h, b + k) - f(a, b) = \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + R.$$

If $f(a, b)$ is a turning value, the expression on the left-hand side of $(E)$ must retain the same sign for all values of $h$ and $k$ sufficiently small in numerical value, the negative sign for a maximum value [$(A)$, p. 246] and the positive sign for a minimum value [$(B)$, p. 246], i.e. $f(a, b)$ will be a maximum or a minimum according as the right-hand side of $(E)$ is negative or positive. Now $R$ is of a degree higher than the second in $h$ and $k$. Hence as $h$ and $k$ diminish in numerical value it seems plausible to conclude that *the numerical value of $R$ will eventually become and remain less than*

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* See *Cours d'Analyse*, Vol. I, by C. Jordan.

*the numerical value of the sum of the three terms of the second degree written down on the right-hand side of* $(E)$.[*] Then the sign of the right-hand side (and therefore also of the left-hand side) will be the same as the sign of the expression

$$(F) \qquad h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}.$$

But from Algebra we know that the quadratic expression

$$h^2A + 2\,hkC + k^2B$$

always has the same sign as $A$ (or $B$) when $AB - C^2 > 0$.

Applying this to $(F)$, $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial y^2}$, $C = \frac{\partial^2 f}{\partial x\partial y}$, and we see that $(F)$, and therefore also the left-hand member of $(E)$, has the same sign as $\frac{\partial^2 f}{\partial x^2}\left(\text{or } \frac{\partial^2 f}{\partial y^2}\right)$ when

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 > 0.$$

Hence the following **rule for finding maximum and minimum values of a function $f(x, y)$.**

**First step.** *Solve the simultaneous equations*

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

**Second step.** *Calculate for these values of $x$ and $y$ the value of*

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2.$$

**Third step.** *The function will have a*

**maximum** *if* $\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2}\left(\text{or } \frac{\partial^2 f}{\partial y^2}\right) < 0$;

**minimum** *if* $\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2}\left(\text{or } \frac{\partial^2 f}{\partial y^2}\right) > 0$;

**neither a maximum nor a minimum** *if* $\Delta < 0$.

**The question is undecided** *if* $\Delta = 0$.[†]

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[*] Peano has shown that this conclusion does not always hold. See the article on "Maxima and Minima of Functions of Several Variables," by Professor James Pierpont in the *Bulletin of the American Mathematical Society*, Vol. IV.

[†] The discussion of the text merely renders the given rule plausible. The student should observe that the case $\Delta = 0$ is omitted in the discussion.

The student should notice that this *rule* does not necessarily give *all* maximum and minimum values. For a pair of values of $x$ and $y$ determined by the *first step* may cause $\Delta$ to vanish, and may lead to a maximum or a minimum or neither. Further investigation is therefore necessary for such values. The rule is, however, sufficient for solving many important examples.

The question of maxima and minima of functions of three or more independent variables must be left to more advanced treatises.

Ex. 1. Examine the function $3axy - x^3 - y^3$ for maximum and minimum values.

*Solution.* $$f(x, y) = 3axy - x^3 - y^3.$$

*First step.* $$\frac{\partial f}{\partial x} = 3ay - 3x^2 = 0, \quad \frac{\partial f}{\partial y} = 3ax - 3y^2 = 0.$$

Solving these two equations simultaneously, we get

$$x = 0, \qquad x = a,$$
$$y = 0; \qquad y = a.$$

*Second step.* $$\frac{\partial^2 f}{\partial x^2} = -6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 3a, \quad \frac{\partial^2 f}{\partial y^2} = -6y;$$

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 36xy - 9a^2.$$

*Third step.* When $x = 0$ and $y = 0$, $\Delta = -9a^2$, and there can be neither a maximum nor a minimum at (0, 0).

When $x = a$ and $y = a$, $\Delta = +27a^2$; and since $\frac{\partial^2 f}{\partial x^2} = -6a$, we have the conditions for a maximum value of the function fulfilled at $(a, a)$. Substituting $x = a$, $y = a$ in the given function, we get its maximum value equal to $a^3$.

Ex. 2. Divide $a$ into three parts such that their product shall be a maximum.

*Solution.* Let $x$ = first part, $y$ = second part; then $a - (x + y) = a - x - y$ = third part, and the function to be examined is

$$f(x, y) = xy(a - x - y).$$

*First step.* $$\frac{\partial f}{\partial x} = ay - 2xy - y^2 = 0, \quad \frac{\partial f}{\partial y} = ax - 2xy - x^2 = 0.$$

Solving simultaneously, we get as one pair of values $x = \frac{a}{3}$, $y = \frac{a}{3}$.*

*Second step.* $$\frac{\partial^2 f}{\partial x^2} = -2y, \quad \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y, \quad \frac{\partial^2 f}{\partial y^2} = -2x;$$

$$\Delta = 4xy - (a - 2x - 2y)^2.$$

---

* $x = 0$, $y = 0$ are not considered, since from the nature of the problem we would then have a minimum.

*Third step.* When $x = \frac{a}{3}$ and $y = \frac{a}{3}$, $\Delta = \frac{a^2}{3}$; and since $\frac{\partial^2 f}{\partial x^2} = -\frac{2a}{3}$, it is seen that our product is a maximum when $x = \frac{a}{3}$, $y = \frac{a}{3}$. Therefore the third part is also $\frac{a}{3}$, and the maximum value of the product is $\frac{a^3}{27}$.

## EXAMPLES

1. Find the maximum value of $x^2 + xy + y^2 - ax - by$. *Ans.* $\frac{1}{3}(ab - a^2 - b^2)$.

2. Show that $\sin x + \sin y + \cos(x + y)$ is a minimum when $x = y = \frac{3\pi}{2}$, and a maximum when $x = y = \frac{\pi}{6}$.

3. Show that $xe^{y + x \sin y}$ has neither a maximum nor a minimum.

4. Show that the maximum value of $\frac{(ax + by + c)^2}{x^2 + y^2 + 1}$ is $a^2 + b^2 + c^2$.

5. Find the greatest rectangular parallelopiped that can be inscribed in an ellipsoid. That is, find the maximum value of $8xyz$ (= volume) subject to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad \textit{Ans. } \frac{8abc}{3\sqrt{3}}.$$

*Hint.* Let $u = xyz$, and substitute the value of $z$ from the equation of the ellipsoid. This gives

$$u^2 = x^2y^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

where $u$ is a function of only two variables.

6. Show that the surface of a rectangular parallelopiped of given volume is least when the solid is a cube.

7. Examine $x^4 + y^4 - x^2 + xy - y^2$ for maximum and minimum values.

*Ans.* Maximum when $x = 0$, $y = 0$;
minimum when $x = y = \pm \frac{1}{2}$, and when $x = -y = \pm \frac{1}{2}\sqrt{3}$.

8. Show that when the radius of the base equals the depth, a steel cylindrical standpipe of a given capacity requires the least amount of material in its construction.

9. Show that the most economical dimensions for a rectangular tank to hold a given volume are a square base and a depth equal to one half the side of the base.

10. The electric time constant of a cylindrical coil of wire is

$$u = \frac{mxyz}{ax + by + cz},$$

where $x$ is the mean radius, $y$ is the difference between the internal and external radii, $z$ is the axial length, and $m$, $a$, $b$, $c$ are known constants. The volume of the coil is $nxyz = g$. Find the values of $x$, $y$, $z$ which make $u$ a minimum if the volume of the coil is fixed.

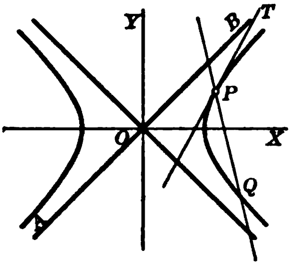
*Ans.* $ax = by = cz = \sqrt[3]{\frac{abcg}{n}}$.

# CHAPTER XXI

## ASYMPTOTES. SINGULAR POINTS. CURVE TRACING

**163. Rectilinear asymptotes.** An *asymptote* to a curve is the limiting position* of a tangent whose point of contact moves off to an infinite distance from the origin.†

Thus, in the hyperbola, the asymptote $AB$ is the limiting position of the tangent $PT$ as the point of contact $P$ moves off to the right to an infinite distance. In the case of algebraic curves the following definition is useful: an asymptote is the limiting position of a secant as two points of intersection of the secant with a branch of the curve move off in the same direction along that branch to an infinite distance. For example, the asymptote $AB$ is the limiting position of the secant $PQ$ as $P$ and $Q$ move upwards to an infinite distance.



**164. Asymptotes found by method of limiting intercepts.** The equation of the tangent to a curve at $(x_1, y_1)$ is by (1), p. 89,

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1).$$

First placing $y = 0$ and solving for $x$, and then placing $x = 0$ and solving for $y$, and denoting the intercepts by $x_i$ and $y_i$ respectively, we get

$$x_i = x_1 - y_1 \frac{dx_1}{dy_1} = \textit{intercept on } OX;$$

$$y_i = y_1 - x_1 \frac{dy_1}{dx_1} = \textit{intercept on } OY.$$

---

* A line that approaches a fixed straight line as a limiting position cannot be wholly at infinity; hence it follows that an asymptote must pass within a finite distance of the origin. It is evident that a curve which has no infinite branch can have no real asymptote.

† Or, less precisely, an asymptote to a curve is sometimes defined as a tangent whose point of contact is at an infinite distance.

Since an asymptote must pass within a finite distance of the origin, one or both of these intercepts must approach finite values as limits when the point of contact $(x_1, y_1)$ moves off to an infinite distance. If

$$\text{limit}(x_i) = a \text{ and } \text{limit}(y_i) = b,$$

then the equation of the asymptote is found by substituting the limiting values $a$ and $b$ in the equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

If only one of these limits exists, but

$$\text{limit}\left(\frac{dy_1}{dx_1}\right) = m,$$

then we have one intercept and the slope given, so that the equation of the asymptote is

$$y = mx + b \text{ or } x = \frac{y}{m} + a.$$

Ex. 1. Find the asymptotes to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

*Solution.* $\frac{dy}{dx} = \frac{b^2x}{a^2y} = \pm \frac{b}{a} \frac{1}{\sqrt{1 - \frac{a^2}{x^2}}}$, and $m = \underset{x = \infty}{\text{limit}}\left(\frac{dy}{dx}\right) = \pm \frac{b}{a}$.

Also, $x_i = \frac{a^2}{x}$ and $y_i = -\frac{b^2}{y}$; hence these intercepts are zero when $x = y = \infty$. Therefore the asymptotes pass through the origin (see figure on p. 252) and their equations are

$$y - 0 = \pm \frac{b}{a}(x - 0), \text{ or, } ay = \pm bx. \quad \textit{Ans.}$$

This method is frequently too complicated to be of practical use. The most convenient method of determining the asymptotes to algebraic curves is given in the next section.

**165. Method for determining asymptotes to algebraic curves.** Given the algebraic equation in two variables,

$$(A) \qquad f(x, y) = 0.$$

If this equation when cleared of fractions and radicals is of degree $n$, then it may be arranged according to descending powers of one of the variables, say $y$, in the form

$$(B)\qquad ay^n + (bx + c)y^{n-1} + (dx^2 + ex + f)y^{n-2} + \cdots = 0.^*$$

For a given value of $x$, this equation determines in general $n$ values of $y$.

Case I. **To determine the asymptotes to the curve (B) which are parallel to the coördinate axis.** Let us first investigate for asymptotes parallel to $OY$. The equation of any such asymptote is of the form

$$(C)\qquad x = k,$$

and it must have two points of intersection with $(B)$ having infinite ordinates.

*First.* Suppose $a$ is not zero in $(B)$, that is, the term in $y^n$ is present. Then for any finite value of $x$, $(B)$ gives $n$ values of $y$, *all finite.* Hence all such lines as $(C)$ will intersect $(B)$ in points having finite ordinates, and *there are no asymptotes parallel to* $OY$.

*Second.* Next suppose $a = 0$ but $b$ and $c$ are not zero. Then we know from Algebra that one root $(= y)$ of $(B)$ is infinite for every finite value of $x$; that is, any arbitrary line $(C)$ intersects $(B)$ at only one point having an infinite ordinate. If now in addition

$$bx + c = 0,\ \text{or},$$

$$(D)\qquad x = -\frac{c}{b},$$

then the first two terms in $(B)$ will drop out, and hence two of its roots are infinite. That is, $(D)$ and $(B)$ intersect in two points having infinite ordinates, and therefore $(D)$ *is the equation of an asymptote to* $(B)$ *which is parallel to* $OY$.

*Third.* If $a = b = c = 0$, there are two values of $x$ that make $y$ in $(B)$ infinite, namely, those satisfying the equation

$$(E)\qquad dx^2 + ex + f = 0.$$

*Solving* $(E)$ *for* $x$, *we get two asymptotes parallel to* $OY$, and so on in general.

---

* For use in this section the attention of the student is called to the following theorem from Algebra: Given an algebraic equation of degree $n$,

$$Ay^n + By^{n-1} + Cy^{n-2} + Dy^{n-3} + \cdots = 0.$$

When $A$ approaches zero, one root (value of $y$) approaches $\infty$.
When $A$ and $B$ approach zero, two roots approach $\infty$.
When $A$, $B$, and $C$ approach zero, three roots approach $\infty$, etc.

In the same way, by arranging $f(x, y)$ according to descending powers of $x$, we may find the asymptotes parallel to $OX$. Hence **the following rule for finding the asymptotes parallel to the coördinate axes:**

**First step.** *Equate to zero the coefficient of the highest power of $x$ in the equation. This gives all asymptotes parallel to $OX$.*

**Second step.** *Equate to zero the coefficient of the highest power of $y$ in the equation. This gives all asymptotes parallel to $OY$.*

NOTE. Of course if one or both of these coefficients do not involve $x$ (or $y$), they cannot be zero, and there will be no corresponding asymptote.

**Ex. 1. Find the asymptotes of the curve $a^2x = y(x-a)^2$.**

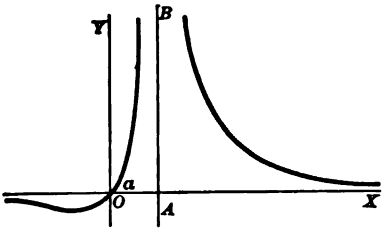
***Solution.*** **Arranging the terms according to powers of $x$,**

$$yx^2 - (2ay + a^2)x + a^2y = 0.$$

**Equating to zero the coefficient of the highest power of $x$, we get $y = 0$ as the asymptote parallel to $OX$. In fact the asymptote coincides with the axis of $x$. Arranging the terms according to powers of $y$,**

$$(x-a)^2y - a^2x = 0.$$

**Placing the coefficient of $y$ equal to zero, we get $x = a$ twice, showing that $AB$ is a double asymptote parallel to $OY$. If this curve is examined for asymptotes oblique to the axes by the method explained below, it will be seen that there are none. Hence $y = 0$ and $x = a$ are the only asymptotes of the given curve.**



CASE II. **To determine asymptotes oblique to the coördinate axes.** Given the algebraic equation

$$(F) \qquad f(x, y) = 0.$$

Consider the straight line

$$(G) \qquad y = mx + k.$$

It is required to determine $m$ and $k$ so that the line $(G)$ shall be an asymptote to the curve $(F)$.

Since an asymptote is the limiting position of a secant as two points of intersection on the same branch of the curve move off to an infinite distance, if we eliminate $y$ between $(F)$ and $(G)$, the resulting equation in $x$, namely,

$$(H) \qquad f(x, mx + k) = 0,$$

must have two infinite roots. But this requires that the coefficients of the two highest powers of $x$ shall vanish. Equating these coefficients to zero, we get two equations from which the required values of $m$ and $k$ may be determined. Substituting these values in ($G$) gives the equation of an asymptote. Hence the following **rule for finding asymptotes oblique to the coördinate axes:**

**First step.** *Replace $y$ by $mx + k$ in the given equation and expand.*

**Second step.** *Arrange the terms according to descending powers of $x$.*

**Third step.** *Equate to zero the coefficients of the two highest powers* of $x$, and solve for $m$ and $k$.*

**Fourth step.** *Substitute these values of $m$ and $k$ in*

$$y = mx + k.$$

*This gives the required asymptotes.*

Ex. 2. Examine $y^3 = 2ax^2 - x^3$ for asymptotes.

*Solution.* Since none of the terms involve both $x$ and $y$, it is evident that there are no asymptotes parallel to the coördinate axes. To find the oblique asymptotes, eliminate $y$ between the given equation and $y = mx + k$. This gives

$$(mx + k)^3 = 2ax^2 - x^3;$$

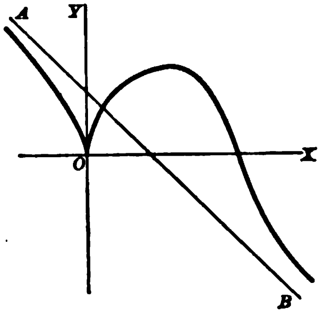
and arranging the terms in powers of $x$,

$$(1 + m^3)x^3 + (3m^2k - 2a)x^2 + 3k^2mx + k^3 = 0.$$

Placing the first two coefficients equal to zero,

$$1 + m^3 = 0 \text{ and } 3m^2k - 2a = 0.$$

Solving, we get $m = -1, k = \frac{2a}{3}\cdot$ Substituting in $y = mx + k$, we have $y = -x + \frac{2a}{3}$, the equation of asymptote $AB$.



EXAMPLES

Examine the first eight curves for asymptotes by the method of § 164, and the remaining ones by the method of § 165.

1. $y = e^x$. *Ans.* $y = 0$.

2. $y = e^{-x^2}$. *Ans.* $y = 0$.

---

* If the term involving $x^{n-1}$ is missing, or if the value of $m$ obtained by placing the first coefficient equal to zero causes the second coefficient to vanish, then by placing the coefficients of $x^n$ and $x^{n-2}$ equal to zero we obtain two equations from which the values of $m$ and $k$ may be found. In this case we shall in general obtain two $k$'s for each $m$, that is, pairs of parallel oblique asymptotes. Similarly, if the term in $x^{n-2}$ is also missing, each value of $m$ furnishes three parallel oblique asymptotes, and so on.

3. $y = \log x$. *Ans.* $x = 0$.

4. $y = \left(1 + \frac{1}{x}\right)^x$. $y = e$.

5. $y = \tan x$. $n$ being any odd integer, $x = \frac{n\pi}{2}$.

6. $y = e^{\frac{1}{x}} - 1$. $x = 0,\ y = 0$.

7. $y^3 = 6x^2 + x^3$. $y = x + 2$.

8. Show that the parabola has no asymptotes.

9. $y^3 = a^3 - x^3$. $y + x = 0$.

10. The cissoid $y^2 = \frac{x^3}{2r - x}$. $x = 2r$.

11. $y^2a = y^2x + x^3$. $x = a$.

12. $y^2(x^2 + 1) = x^2(x^2 - 1)$. $y = \pm x$.

13. $y^2(x - 2a) = x^3 - a^3$. $x = 2a,\ y = \pm(x + a)$.

14. $x^2y^2 = a^2(x^2 + y^2)$. $x = \pm a,\ y = \pm a$.

15. $y(x^2 - 3bx + 2b^2) = x^3 - 3ax^2 + a^3$. $x = b,\ x = 2b,\ y + 3a = x + 3b$.

16. $y = c + \frac{a^3}{(x - b)^2}$. $y = c,\ x = b$.

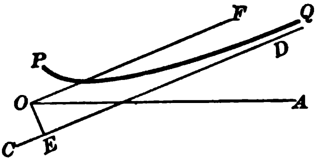
17. The folium $x^3 + y^3 - 3axy = 0$. $y + x + a = 0$.

18. The witch $x^2y = 4a^2(2a - y)$. $y = 0$.

19. $xy^2 + x^2y = a^3$. $x = 0,\ y = 0,\ x + y = 0$.

20. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y = 1$.
$x + 2y = 0,\ x + y = 1,\ x - y = -1$.

**166. Asymptotes in polar coördinates.** Let $f(\rho, \theta) = 0$ be the equation of the curve $PQ$ having the asymptote $CD$. As the asymptote must pass within a finite distance (as $OE$) of the origin, and the point of contact is at an infinite distance, it is evident that the radius vector $OF$ drawn to the point of contact is parallel to the asymptote, and the subtangent $OE$ is perpendicular to it. Or, more precisely, the distance of the asymptote from the origin is the limiting value of the polar subtangent as the point of contact moves off an infinite distance.



**To determine the asymptotes to a polar curve, proceed as follows:**

**First step.** *Find from the equation of the curve the values of* $\theta$ *which make* $\rho = \infty$.* *These values of* $\theta$ *give the directions of the asymptotes.*

**Second step.** *Find the limit of the polar subtangent*

$$\rho^2 \frac{d\theta}{d\rho}, \qquad (7), \text{ p. } 99$$

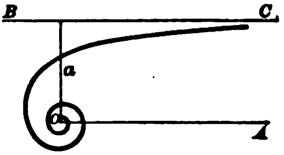
*as* $\theta$ *approaches each such value, remembering that* $\rho$ *approaches* $\infty$ *at the same time.*

**Third step.** *If the limiting value of the polar subtangent is finite, there is a corresponding asymptote at that distance from the origin and parallel to the radius vector drawn to the point of contact. When this limit is positive the asymptote is to the right, and when negative, to the left of the origin, looking in the direction of the infinite radius vector.*

## EXAMPLES

1. Examine the hyperbolic spiral $\rho = \frac{a}{\theta}$ for asymptotes.

*Solution.* When $\theta = 0$, $\rho = \infty$. Also $\frac{d\rho}{\theta d} = -\frac{a}{\theta^2}$, hence



$$\text{Subtangent} = \rho^2 \frac{d\theta}{d\rho} = \frac{a^2}{\theta^2} \cdot -\frac{\theta^2}{a^2} = -a.$$

$$\therefore \underset{\theta = 0}{\text{limit}} \left[\rho^2 \frac{d\theta}{d\rho}\right] = -a, \text{ which is finite.}$$

It happens in this case that the subtangent is the same for all values of $\theta$. The curve has therefore an asymptote $BC$ parallel to the initial line $OA$ and at a distance $a$ above it.

Examine the following curves for asymptotes.

2. $\rho \cos \theta = a \cos 2\theta$.
*Ans.* There is an asymptote perpendicular to the initial line at a distance $a$ to the left of the origin.

3. $\rho = a \tan \theta$.
*Ans.* There are two asymptotes perpendicular to the initial line and at a distance $a$ from the origin, on either side of it.

4. The lituus $\rho\theta^{\frac{1}{2}} = a$. *Ans.* The initial line.

---

* If the equation can be written as a polynomial in $\rho$, these values of $\theta$ may be found by equating to zero the coefficient of the highest power of $x$ (see footnote, p. 254).

5. $\rho = a \sec 2\theta$.

*Ans.* There are four asymptotes at the same distance $\frac{a}{2}$ from the origin, and inclined 45° to the initial line.

6. $(\rho - a)\sin\theta = b$.

*Ans.* There is an asymptote parallel to the initial line at the distance $b$ above it.

7. $\rho = a(\sec 2\theta + \tan 2\theta)$.

*Ans.* There are two asymptotes parallel to $\theta = \frac{\pi}{4}$, at the distance $a$ on each side of the origin.

8. Show that the initial line is an asymptote to two branches of the curve $\rho^2 \sin\theta = a^2 \cos 2\theta$.

9. Parabola $\rho = \frac{a}{1 - \cos\theta}$. *Ans.* There is no asymptote.

**167. Singular points.** Given a curve whose equation is

$$f(x, y) = 0.$$

Any point on the curve for which

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

is called a ***singular point*** of the curve. All other points are called *ordinary points* of the curve. Since by **(55 *a*)**, p. 202, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}},$$

it is evident that at a singular point the direction of the curve (or tangent) is indeterminate, for the slope takes the form $\frac{0}{0}$. In the next section it will be shown how tangents at such points may be found.

**168. Determination of the tangent to an algebraic curve at a given point by inspection.** If we transform the given equation to a new set of parallel coördinate axes having as origin the point in question on the curve, we know that the new equation will have no constant term. Hence it may be written in the form

$$(A) \qquad f(x, y) = ax + by + (cx^2 + dxy + ey^2) + (fx^3 + gx^2y + hxy^2 + iy^3) + \cdots = 0.$$

The equation of a tangent to the curve at the given point (now the origin) will be

(*B*) $$y = \left(\frac{dy}{dx}\right)x.$$ By (1), p. 89

Let $y = mx$ (by 54 (c), p. 3) be the equation of a line through the origin $O$ and a second point $P$ on the locus of (*A*). If then $P$ approaches $O$ along the curve, we have from (*B*)

(*C*) $$\text{limit } m = \frac{dy}{dx}.$$

Let $O$ be an ordinary point. Then, by § 167, $a$ and $b$ do not both vanish since at (0, 0), from (*A*),

$$\frac{\partial f}{\partial x} = a,\ \frac{\partial f}{\partial y} = b.$$

Replace $y$ in (*A*) by $mx$, divide out the factor $x$, and let $x$ approach zero as a limit. Then (*A*) will become*

$$a + bm = 0.$$

Hence we have from (*B*) and (*C*)

$$ax + by = 0,$$

the equation of the tangent. The left-hand member is seen to consist of the terms of the first degree in (*A*).

When $O$ is not an ordinary point we have $a = b = 0$. Assume that $c$, $d$, $e$ do not all vanish. Then proceeding as before (except that we divide out the factor $x^2$), we find, after letting $x$ approach the limit zero, that (*A*) becomes

$$c + dm + em^2 = 0,$$

or, from (*C*),

(*D*) $$c + d\left(\frac{dy}{dx}\right) + e\left(\frac{dy}{dx}\right)^2 = 0.$$

---

* After dividing by $x$ an algebraic equation in $m$ remains whose coefficients are functions of $x$. If now $x$ approaches zero as a limit, the theorem holds that one root of this equation in $m$ will approach the limit $-a \div b$.

Substituting from (*B*), we see that

$$(E) \qquad cx^2 + dxy + ey^2 = 0$$

is the equation of the pair of tangents at the origin. The left-hand member is seen to consist of the terms of the second degree in (*A*). Such a singular point of the curve is called a *double point* from the fact that there are two tangents to the curve at that point.

Since at (0, 0), from (*A*),

$$\frac{\partial^2 f}{\partial x^2} = 2\,c, \quad \frac{\partial^2 f}{\partial x \partial y} = d, \quad \frac{\partial^2 f}{\partial y^2} = 2\,e,$$

it is evident that (*D*) may be written in the form

$$(F) \qquad \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y}\left(\frac{dy}{dx}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

In the same manner, if

$$a = b = c = d = e = 0,$$

there is a *triple point* at the origin, the equation of the three tangents being

$$fx^3 + gx^2y + hxy^2 + iy^3 = 0.$$

And so on in general.

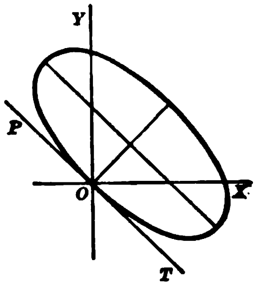
If we wish to investigate the appearance of a curve at a given point, it is of fundamental importance to solve the tangent problem for that point. The above results indicate that this can be done *by simple inspection* after we have transformed the origin to that point.

Hence we have the following **rule for finding the tangents at a given point.**

**First step.** *Transform the origin to the point in question.*

**Second step.** *Arrange the terms of the resulting equation according to ascending powers of x and y.*

**Third step.** *Set the group of terms of lowest degree equal to zero. This gives the equation of the tangents at the point (origin).*



**Ex. 1.** Find the equation of the tangent to the ellipse

$$5x^2+5y^2+2xy-12x-12y=0$$

at the origin.

*Solution.* Placing the terms of lowest (first) degree equal to zero, we get

$$-12x-12y=0,$$

or, $$x+y=0,$$

which is then the equation of the tangent $PT$ at the origin.

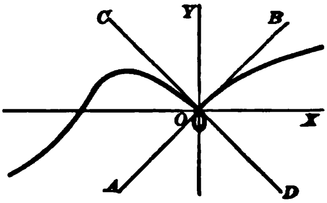
**Ex. 2.** Examine the curve $3x^2-xy-2y^2+x^3-8y^3=0$ for tangents at the origin.

*Solution.* Placing the terms of lowest (second) degree equal to zero,

$$3x^2-xy-2y^2=0,$$

or, $$(x-y)(3x+2y)=0,$$

$x-y=0$ being the equation of the tangent $AB$, and $3x+2y=0$ the equation of the tangent $CD$. The origin is, then, a double point of the curve.



Since the roots of the quadratic equation (*F*), p. 261, namely,

$$\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+2\frac{\partial^2 f}{\partial x\partial y}\left(\frac{dy}{dx}\right)+\frac{\partial^2 f}{\partial x^2}=0,$$

may be real and unequal, real and equal, or imaginary, there are three cases of double points to be considered, according as

$$(G)\qquad \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}$$

is positive, zero, or negative (see 3, p. 1).

### 169. Nodes. $\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}>0.$

In this case there are two real and unequal values of the slope $\left(=\frac{dy}{dx}\right)$ found from (*F*), so that we have two distinct real tangents to the curve at the singular point in question. This means that the curve passes through the point in two different directions, or, in other words, two branches of the curve cross at this point. Such a singular point we call a *real double point* of the curve, or a *node*. **Hence the conditions to be satisfied at a node are**

$$f(x,y)=0,\ \frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0,\ \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}>0.$$

**Ex. 1. Examine the lemniscate $y^2 = x^2 - x^4$ for singular points.**

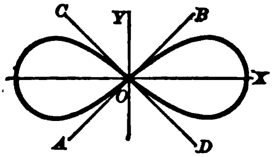
*Solution.* Here
$$f(x, y) = y^2 - x^2 + x^4 = 0.$$

Also,
$$\frac{\partial f}{\partial x} = -2x + 4x^3 = 0, \quad \frac{\partial f}{\partial y} = 2y = 0.$$

The point (0, 0) is a singular point since its coördinates satisfy the above three equations. We have at (0, 0),

$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 2.$$

$$\therefore \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = 4,$$



and the origin is a double point (node) through which two branches of the curve pass in different directions. By placing the terms of the lowest (second) degree equal to zero we get

$$y^2 - x^2 = 0, \text{ or } y = x \text{ and } y = -x,$$

the equations of the two tangents $AB$ and $CD$ at the singular point or node (0, 0).

**170. Cusps.** $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = 0.$

In this case there are two real and equal values of the slope found from ($F$), hence there are two coincident tangents. This means that the two branches of the curve which pass through the point are tangent. When the curve recedes from the tangent in both directions from the point of tangency, the singular point is called a *point of osculation*; if it recedes from the point of tangency in one direction only, it is called a *cusp*. There are two kinds of cusps.

*First kind.* When the two branches lie on opposite sides of the common tangent.

*Second kind.* When the two branches lie on the same side of the common tangent.*

The following examples illustrate how we may determine the nature of singular points coming under this head.

**Ex. 1. Examine $a^4y^2 = a^2x^4 - x^6$ for singular points.**

*Solution.* Here $f(x, y) = a^4y^2 - a^2x^4 + x^6 = 0,$

$$\frac{\partial f}{\partial x} = -4a^2x^3 + 6x^5 = 0, \quad \frac{\partial f}{\partial y} = 2a^4y = 0,$$

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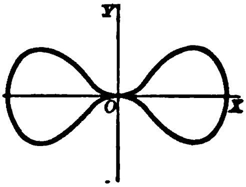
* Meaning in the neighborhood of the singular point.

and (0, 0) is a singular point since it satisfies the above three equations. Also, at (0, 0) we have

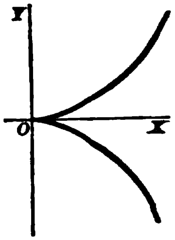


$$\frac{\partial^2 f}{\partial x^2}=0,\ \frac{\partial^2 f}{\partial x \partial y}=0,\ \frac{\partial^2 f}{\partial y^2}=2a^4.$$

$$\therefore \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}=0;$$

and since the curve is symmetrical with respect to $OY$, the origin is a point of osculation. Placing the terms of lowest (second) degree equal to zero, we get $y^2=0$, showing that the two common tangents coincide with $OX$.

Ex. 2. Examine $y^2 = x^3$ for singular points.



*Solution.* Here $f(x, y) = y^2 - x^3 = 0$,

$$\frac{\partial f}{\partial x}=-3x^2=0,\ \frac{\partial f}{\partial y}=2y=0,$$

showing that (0, 0) is a singular point. Also, at (0, 0) we have

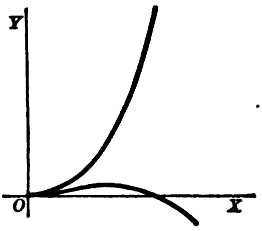
$$\frac{\partial^2 f}{\partial x^2}=0,\ \frac{\partial^2 f}{\partial x \partial y}=0,\ \frac{\partial^2 f}{\partial y^2}=2.\ \therefore \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}=0.$$

This is not a point of osculation, however, for if we solve the given equation for $y$, we get

$$y=\pm\sqrt{x^3},$$

which shows that the curve extends to the right only of $OY$, for negative values of $x$ make $y$ imaginary. The origin is therefore a cusp, and since the branches lie on opposite sides of the common tangent it is a cusp of the first kind. Placing the terms of lowest (second) degree equal to zero, we get $y^2 = 0$, showing that the two common tangents coincide with $OX$.

Ex. 3. Examine $(y - x^2)^2 = x^5$ for singular points.



*Solution.* Proceeding as in the last example, we find a cusp at (0, 0), the common tangents to the two branches coinciding with $OX$. Solving for $y$,

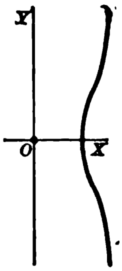
$$y = x^2 \pm x^{\frac{5}{2}}.$$

If we let $x$ take on any value between 0 and 1, $y$ takes on two different positive values, showing that in the vicinity of the origin both branches lie above the common tangent. Hence the singular point (0, 0) is a cusp of the second kind.

## 171. Conjugate or isolated points. $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}<0.$

In this case the values of the slope found from (*B*) are imaginary. Hence there are no real tangents; the singular point is the real intersection of imaginary branches of the curve, and the coördinates of no other real point in the immediate vicinity satisfy the equation of the curve. Such an isolated point is called a *conjugate point.*

**Ex. 1. Examine the curve $y^2 = x^3 - x^2$ for singular points.**

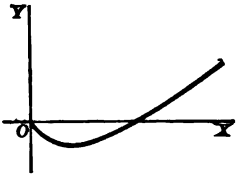


***Solution.*** **Here (0, 0) is found to be a singular point of the curve at which $\frac{dy}{dx} = \pm\sqrt{-1}$. Hence the origin is a conjugate point. Solving the equation for $y$,**

$$y = \pm x\sqrt{x-1}.$$

**This shows clearly that the origin is an isolated point of the curve, for no values of $x$ between 0 and 1 give real values of $y$.**

**172. Transcendental singularities.** A curve whose equation involves transcendental functions is called a transcendental curve. Such a curve may have an *end point*, at which it terminates abruptly, caused by a discontinuity in the function; or a *salient point* at which two branches of the curve terminate without having a common tangent, caused by a discontinuity in the derivative.



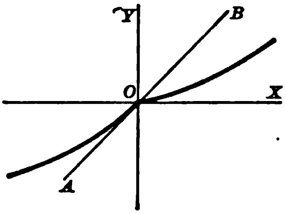
**Ex. 1. Show that $y = x \log x$ has an end point at the origin.**

***Solution.*** **$x$ cannot be negative since negative numbers have no logarithms; hence the curve extends only to the right of $OY$. When $x = 0$, $y = 0$. There being only one value of $y$ for each positive value of $x$, the curve consists of a single branch terminating at the origin, which is therefore an end point.**

**Ex. 2. Show that $y = \frac{x}{1+e^{\frac{1}{x}}}$ has a salient point at the origin.**

***Solution.*** **Here**

$$\frac{dy}{dx} = \frac{1}{1+e^{\frac{1}{x}}} + \frac{e^{\frac{1}{x}}}{x\left(1+e^{\frac{1}{x}}\right)^2}.$$



**If $x$ is positive and approaches zero as a limit, we have ultimately**

$$y = 0 \text{ and } \frac{dy}{dx} = 0.$$

**If $x$ is negative and approaches zero as a limit, we get ultimately**

$$y = 0 \text{ and } \frac{dy}{dx} = 1.$$

**Hence at the origin two branches meet, one having $OX$ as its tangent and the other, $AB$, making an angle of 45° with $OX$.**

## EXAMPLES

1. Show that $y^2 = 2x^2 + x^3$ has a node at the origin, the slopes of the tangents being $\pm\sqrt{2}$.

2. Show that the origin is a node of $y^2(a^2 + x^2) = x^2(a^2 - x^2)$, and that the tangents bisect the angles between the axes.

3. Prove that $(a, 0)$ is a node of $y^2 = x(x - a)^2$, and that the slopes of the tangents are $\pm\sqrt{a}$.

4. Prove that $a^3y^2 - 2abx^2y - x^5 = 0$ has a point of osculation at the origin.

5. Show that the curve $y^2 = x^5 + x^4$ has a point of osculation at the origin.

6. Show that the cissoid $y^2 = \dfrac{x^3}{2a - x}$ has a cusp of the first kind at the origin.

7. Show that $y^3 = 2ax^2 - x^3$ has a cusp of the first kind at the origin.

8. In the curve $(y - x^2)^2 = x^n$ show that the origin is a cusp of the first or second kind according as $n$ is $<$ or $>$ 4.

9. Prove that the curve $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$ has a cusp of the second kind at the origin.

10. Show that the origin is a conjugate point on the curve $y^2(x^2 - a^2) = x^4$.

11. Show that the curve $y^2 = x(a + x)^2$ has a conjugate point at $(-a, 0)$.

12. Show that the origin is a conjugate point on the curve $ay^2 - x^3 + bx^2 = 0$ when $a$ and $b$ have the same sign, and a node when they have opposite signs.

13. Show that the curve $x^4 + 2ax^2y - ay^3 = 0$ has a triple point at the origin, and that the slopes of the tangents are $0, +\sqrt{2}$, and $-\sqrt{2}$.

14. Show that the points of intersection of the curve $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$ with the axes are cusps of the first kind.

15. Show that no curve of the second or third degree in $x$ and $y$ can have a cusp of the second kind.

16. Show that $y = e^x$ has an end point at the origin.

17. Show that $y = x \operatorname{arc} \tan \dfrac{1}{x}$ has a salient point at the origin, the slopes of the tangents being $\pm\dfrac{\pi}{2}$.

**173. Curve tracing.** The elementary method of tracing (or plotting) a curve whose equation is given in rectangular coordinates, and one with which the student is already familiar, is to solve its equation for $y$ (or $x$), assume arbitrary values of $x$ (or $y$), calculate the corresponding values of $y$ (or $x$), plot the respective points, and draw a smooth curve through them, the result being an approximation to the required curve. This process is laborious at best, and in case the equation of the curve is of a degree higher than the second, the solved form of such an equation may be unsuitable for the purpose of computation, or else it may fail altogether, since it is not always possible to solve the equation for $y$ or $x$.

The general form of a curve is usually all that is desired, and very often we care to examine the curve in the neighborhood of a certain point only. To attain this object it is as a rule only necessary to determine some of the important points, lines, and properties of the curve as enumerated below.

No rules for tracing a curve can be given that will apply in all cases, but the student will find it to his advantage to use the following general directions as a guide and to study carefully the examples that are worked out in detail.

**174. General directions for tracing a curve whose equation is given in rectangular coördinates.**

1. *Examine the curve for symmetry.*

(a) If the equation is unchanged when $y$ is replaced by $-y$, the curve is symmetrical with respect to $OX$.

(b) If the equation is unchanged when $x$ is replaced by $-x$, the curve is symmetrical with respect to $OY$.

(c) If the equation is unchanged when $x$ is replaced by $-x$, and $y$ by $-y$, the curve is symmetrical with respect to the origin which is also the center of the curve.

2. *Examine the curve for important points.*

(d) If the equation is satisfied by $x=0$, $y=0$, the curve passes through the origin.

(e) Placing $x=0$ and solving for $y$ gives the intercepts on $OY$. Placing $y=0$ and solving for $x$ gives the intercepts on $OX$.

(f) Find $\frac{dy}{dx}$; this gives the direction of the curve at any point and serves to locate maximum and minimum points (§ 94, p. 120).

(g) Find $\frac{d^2y}{dx^2}$; this gives the direction of curvature at any point and serves to find the points of inflection (§ 98, p. 137).

(h) Examine the curve for singular points (p. 259).

3. (i) *Examine the curve for asymptotes* (§§ 164, 165, p. 252). Determine on which side of each asymptote the corresponding infinite branch lies.

4. (j) *Locate additional points on the curve.* If possible, compute a sufficient number of points on the curve by the elementary method (§ 173, p. 266) to give a fair idea of the locus, and sketch the curve through the points.

Ex. 1. Trace the curve $y^2 = x^3$.

*Solution.* Let us examine the curve in the above order.

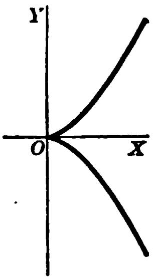
(a) The curve is symmetrical with respect to $OX$.

(b) The curve is not symmetrical with respect to $OY$.

(c) The curve is not symmetrical with respect to the origin.

(d) It passes through the origin.

(e) Its intercepts on the axes are both zero.



(f) $\dfrac{dy}{dx} = \dfrac{3x^2}{2y}$; showing that above $OX$ the curve always has a positive slope, and below $OX$ a negative slope.

It has no maximum or minimum points.

(g) $\dfrac{d^2y}{dx^2} = \pm\dfrac{3}{4\sqrt{x}}$; hence above $OX$ the curve is concave upwards and below $OX$ concave downwards. There are no points of inflection.

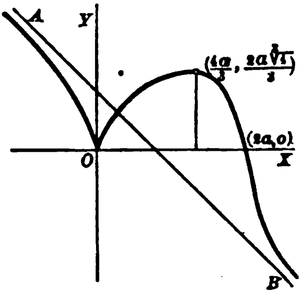
(h) The curve has a cusp of the first kind at the origin, the common tangent coinciding with $OX$.

(i) There are no asymptotes.

(j) $y = \pm\sqrt{x^3}$; hence the curve does not extend to the left of $OY$, since negative values of $x$ make $y$ imaginary. When $x = \infty$, $y = \pm\infty$, showing that there are two infinite branches, one on each side of $OX$. Plotting a number of points and sketching in the curve, we get the semicubical parabola shown in the figure.

Ex. 2. Trace the curve $y^3 = 2ax^2 - x^3$.

*Solution.* This curve is found to be not symmetrical with respect to either axis or the origin, but it passes through the origin and in addition has the intercept $2a$ on $OX$.



$$\frac{dy}{dx} = \frac{4ax - 3x^2}{3y^2};$$

hence when $x = \dfrac{4a}{3}$ the curve has the maximum ordinate $\frac{2}{3}a\sqrt[3]{4}$.

$$\frac{d^2y}{dx^2} = -\frac{8a^2}{9x^{\frac{4}{3}}(2a-x)^{\frac{5}{3}}};$$

hence $x = 2a$ gives a point of inflection on $OX$, to the left of which the curve is concave downwards and to the right concave upwards. The curve has a cusp of the first kind at the origin, the common tangent coinciding with $OY$. The only asymptote is ($AB$ in figure)

$$x + y = \tfrac{2}{3}a,$$

which lies to the right of the infinite branch in the second quadrant and to the left of the infinite branch in the fourth quadrant. From $y = \sqrt[3]{2ax^2 - x^3}$ we plot additional points and draw the curve shown in the figure.

**175. Tracing of curves given by equations in polar coördinates.** The rudimentary method is to solve the equation for $\rho$ when possible, assume values for $\theta$, calculate the corresponding values of $\rho$, plot the points thus found, and draw a curve through them.[*] This work may be facilitated by examining the curve for asymptotes and by noting the values of $\theta$ which make $\rho$ a maximum or a minimum.

Ex. 1. Trace the curve $\rho = 10 \sin 2\theta$.

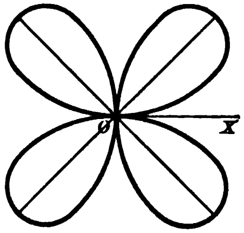
*Solution.* Tabulating the corresponding value of $\theta$ and $\rho$ for every 15° (see table, p. 4), we have

$$\rho = 10 \sin 2\theta.$$

| $\theta$ | $\sin 2\theta$ | $\rho$ |
|---|---|---|
| 0° | 0 | 0 |
| 15° | .50 | 5.0 |
| 30° | .87 | 8.7 |
| 45° | 1.00 | 10.0 |
| 60° | .87 | 8.7 |
| 75° | .50 | 5.0 |
| 90° | 0 | 0 |
| 105° | − .50 | − 5.0 |
| 120° | − .87 | − 8.7 |
| 135° | − 1.00 | − 10.0 |
| 150° | − .87 | − 8.7 |
| 165° | − .50 | − 5.0 |
| 180° | 0 | 0 |
| 195° | .50 | 5.0 |
| 210° | .87 | 8.7 |
| 225° | 1.00 | 10.0 |
| 240° | .87 | 8.7 |
| 255° | .50 | 5.0 |
| 270° | 0 | 0 |
| 285° | − .50 | − 5.0 |
| 300° | − .87 | − 8.7 |
| 315° | − 1.00 | − 10.0 |
| 330° | − .87 | − 8.7 |
| 345° | − .50 | − 5.0 |
| 360° | 0 | 0 |

$\rho$ is a maximum when $\sin 2\theta$ is a maximum, and this occurs when $\sin 2\theta = 1$, or $\theta = 45°$, 225°, etc. This maximum value of $\rho$ is then 10.

$\rho$ is a minimum when $\sin 2\theta$ is a minimum, i.e. when $\sin 2\theta = -1$ or $\theta = 135°$, 315°, etc. Hence the minimum value of $\rho$ is $-10$. When $\theta = 0$, 180°, etc., $\rho = 0$. If we in addition remember that $\sin 2\theta$ is a periodic continuous function of $\theta$, it is not necessary to tabulate many values of $\theta$ and $\rho$. The curve consists of four loops, as shown in the figure, and it is for this reason sometimes called a four-leaved rose.



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[*] The author has designed plotting paper for polar coördinates on which concentric circles and radial lines are drawn in faint blue ink. This paper is desirable for the rapid and accurate plotting of polar curves. Published by the Yale Coöperative Corporation, New Haven, Conn.

## EXAMPLES

Trace the following curves.

1. $y^2(2a - x) = x^3$.
2. $(x^2 + 4a^2)y = 8a^3$.
3. $ay^2 = x^3 - bx^2$.
4. $(y - x)^2 = x^3$.
5. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
6. $x^2y^2 = (b^2 - y^2)(a + y)^2$.
7. $y = \log x$.
8. $y = e^{-x^2}$.
9. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.
10. $y = (x^2 - 1)^2$.
11. $y = \sin x$.
12. $y = \tan x$.
13. $x^2(y - a) = a^3 - xy^2$.
14. $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$
15. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
16. $\rho = a \cos 2\theta$.
17. $\rho = a \sin 3\theta$.
18. $\rho = a(1 - \cos\theta)$.
19. $\rho = a \sin^3 \frac{\theta}{3}$.
20. $\rho = a \sec^2 \frac{\theta}{3}$.

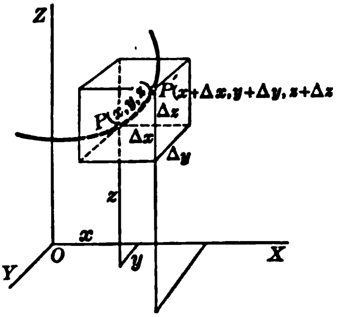
# CHAPTER XXII

## APPLICATIONS TO GEOMETRY OF SPACE

**176. Tangent line and normal plane to a skew curve whose equations are given in parametric form.** The student is already familiar with the parametric representation of a plane curve. To extend this notion to curves in space, let the coördinates of any point $(x, y, z)$ on a skew curve be given as functions of some fourth variable $t$, thus,

$$(A) \qquad x = \phi(t), \quad y = \psi(t), \quad z = \chi(t).$$

The elimination of the parameter $t$ between these equations two by two gives the projecting cylinders of the curve on the coördinate planes.



Let the point $P(x, y, z)$ correspond to the value $t$ of the parameter, and the point $P'(x + \Delta x, y + \Delta y, z + \Delta z)$ correspond to the value $t + \Delta t$; $\Delta x$, $\Delta y$, $\Delta z$ being the increments of $x, y, z$ due to the increment $\Delta t$ as found from equations $(A)$. From Analytic Geometry we know that the direction cosines of the secant (diagonal) $PP'$ are proportional to

$$\Delta x, \ \Delta y, \ \Delta z;$$

or, dividing through by $\Delta t$ and denoting the direction angles of the secant by $\alpha', \beta', \gamma'$,

$$\cos \alpha' : \cos \beta' : \cos \gamma' :: \frac{\Delta x}{\Delta t} : \frac{\Delta y}{\Delta t} : \frac{\Delta z}{\Delta t}.$$

Now let $P'$ approach $P$ along the curve. Then $\Delta t$, and therefore also $\Delta x$, $\Delta y$, $\Delta z$, will approach zero as a limit, the secant $PP'$ will approach the tangent line to the curve at $P$ as a limiting position, and we shall have

$$\cos \alpha : \cos \beta : \cos \gamma :: \frac{dx}{dt} : \frac{dy}{dt} : \frac{dz}{dt},$$

where $\alpha$, $\beta$, $\gamma$ are the direction angles of the tangent (or curve) at $P$. Hence *the equations of the tangent line to the curve*

$$x = \phi(t),\ y = \psi(t),\ z = \chi(t)$$

*at the point* $(x, y, z)$ *are given by*

$$\frac{X - x}{\frac{dx}{dt}} = \frac{Y - y}{\frac{dy}{dt}} = \frac{Z - z}{\frac{dz}{dt}}; \tag{67}$$

*and the equation of the normal plane, i.e. the plane passing through* $(x, y, z)$ *perpendicular to the tangent, is*

$$\frac{dx}{dt}(X - x) + \frac{dy}{dt}(Y - y) + \frac{dz}{dt}(Z - z) = 0, \tag{68}$$

*$X$, $Y$, $Z$ being the variable coördinates.*

Ex. 1. Find the equations of the tangent and the equation of the normal plane to the helix* ($\theta$ being the parameter),

$$\begin{cases} x = a\cos\theta, \\ y = a\sin\theta, \\ z = b\theta, \end{cases}$$

(a) at any point; (b) when $\theta = 2\pi$.

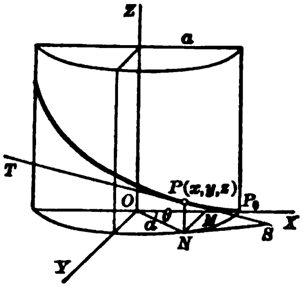
*Solution.* $\quad \dfrac{dx}{d\theta} = -a\sin\theta = -y,\ \dfrac{dy}{d\theta} = a\cos t = x,\ \dfrac{dz}{d\theta} = b.$

Substituting in (67) and (68), we get at $(x, y, z)$,

$$\frac{X - x}{-y} = \frac{Y - y}{x} = \frac{Z - z}{b},\ \text{tangent line};$$

and $\quad -y(X - x) + x(Y - y) + b(Z - z) = 0,$

normal plane.



When $\theta = 2\pi$, the point of contact is $(a, 0, 2b\pi)$, giving $\quad \dfrac{X - a}{0} = \dfrac{Y - 0}{a} = \dfrac{Z - 2b\pi}{b},$

or, $\quad X = a,\ bY = aZ - 2ab\pi,$

the equations of the tangent line; and

$$aY + bZ - 2b^2\pi = 0,$$

the equation of the normal plane.

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[*] The helix may be defined as a curve traced on a right circular cylinder so as to cut all the elements at the same angle.

Take $OZ$ as the axis of the cylinder and the point of starting in $OX$ at $P_0$. Let $a$ = radius of base of cylinder and $\theta$ = angle of rotation. By definition,

$$\frac{PN}{SN} = \frac{PN}{\text{arc } P_0N} = \frac{z}{a\theta} = k\ (\text{const.}),\ \text{or},\ z = ak\theta.$$

Let $ak = b$; then $z = b\theta$. Also, $y = MN = a\sin\theta$, $x = OM = a\cos\theta$.

**177. Tangent plane to a surface.** A straight line is said to be *tangent to a surface* at a point $P$ if it is the limiting position of a secant through $P$ and a neighboring point $P'$ on the surface, when $P'$ is made to approach $P$ along the surface. We now proceed to establish a theorem of fundamental importance.

**Theorem.** *All tangent lines to a surface at a given point*[*] *lie in general in a plane called the tangent plane at that point.*

*Proof.* Let

$$(A) \qquad F(x, y, z) = 0$$

be the equation of the given surface, and let $P(x, y, z)$ be the given point on the surface. If now $P'$ be made to approach $P$ along a curve $C$ lying on the surface and passing through $P$ and $P'$, then evidently the secant $PP'$ approaches the position of a tangent to the curve $C$ at $P$. Now let the equations of the curve $C$ be

$$(B) \qquad x = \phi(t), \; y = \psi(t), \; z = \chi(t).$$

Then the equation $(A)$ must be satisfied identically by these values, and since the total differential of $(A)$ when $x$, $y$, $z$ are defined by $(B)$ must vanish, we have

$$(C) \qquad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0. \qquad \text{(50), p. 199}$$

This equation shows that the tangent line to $C$, whose direction cosines are proportional to

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \qquad \text{p. 271}$$

is perpendicular[†] to a line whose direction cosines are determined by the ratios

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z};$$

and since $C$ is any curve on the surface through $P$, it follows at

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[*] The point in question is assumed to be an ordinary (non-singular) point of the surface, i.e. $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ are not all zero at the point.

[†] From Solid Analytic Geometry we know that if two lines having the direction cosines $\cos\alpha_1, \cos\beta_1, \cos\gamma_1$ and $\cos\alpha_2, \cos\beta_2, \cos\gamma_2$ are perpendicular, then

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0.$$

once, if we replace the point $P(x, y, z)$ by $P_1$ $(x_1, y_1, z_1)$, that all tangent lines to the surface at $P_1$ lie in the plane*

$$\frac{\partial F_1}{\partial x_1}(x - x_1) + \frac{\partial F_1}{\partial y_1}(y - y_1) + \frac{\partial F_1}{\partial z_1}(z - z_1) = 0, \dagger \tag{69}$$

which is then *the formula for finding the equation of a plane tangent at* $(x_1, y_1, z_1)$ *to a surface whose equation is given in the form*

$$F(x, y, z) = 0.$$

In case the equation of the surface is given in the form $z = f(x, y)$, let

$$F(x, y, z) = f(x, y) - z = 0. \tag{D}$$

Then $$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}, \quad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}, \quad \frac{\partial F}{\partial z} = -1.$$

If we evaluate these at $(x_1, y_1, z_1)$ and substitute in (69), we get

$$\frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1) - (z - z_1) = 0, \tag{70}$$

which is then *the formula for finding the equation of a plane tangent at* $(x_1, y_1, z_1)$ *to a surface whose equation is given in the form* $z = f(x, y)$.

In § 137, p. 200, we found (53), the total differential of a function $u$ (or $z$) of $x$ and $y$, namely,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \tag{E}$$

We have now a means of interpreting this result geometrically. For the tangent plane to the surface $z = f(x, y)$ at $(x, y, z)$ is, from (70),

$$Z - z = \frac{\partial z}{\partial x}(X - x) + \frac{\partial z}{\partial y}(Y - y), \tag{F}$$

$X$, $Y$, $Z$ denoting the variable coördinates of any point on the plane. If we substitute

$$X = x + dx \text{ and } Y = y + dy$$

in ($F$), there results

$$Z - z = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \tag{G}$$

Comparing ($E$) and ($G$), we get

$$dz = Z - z. \tag{H}$$ Hence

---

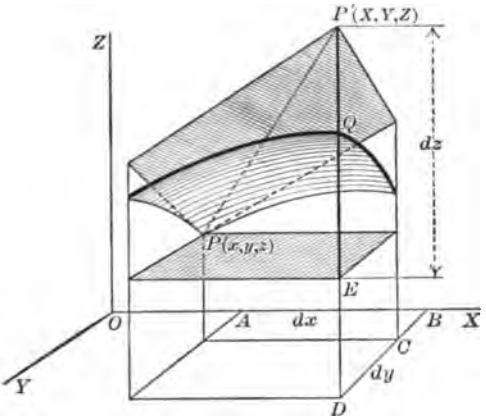
* The direction cosines of the normal to the plane (69) are proportional to $\frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial y_1}, \frac{\partial F_1}{\partial z_1}$. Hence from Analytic Geometry we see that ($C$) is the condition that the tangents whose direction cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are perpendicular to the normal; i.e. the tangents must lie in the plane.

† In agreement with our former practice,

$$\frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial y_1}, \frac{\partial F_1}{\partial z_1}, \frac{\partial z_1}{\partial x_1}, \frac{\partial z_1}{\partial y_1}$$

denote the values of the partial derivatives at the point $(x_1, y_1, z_1)$.

**Theorem.** *The total differential of a function $f(x, y)$ corresponding to the increments $dx$ and $dy$ equals the corresponding increment of the $z$ coördinate of the tangent plane to the surface $z = f(x, y)$.*



Thus, in the figure, $PP'$ is the plane tangent to surface $PQ$ at $P(x, y, z)$.

Let $\qquad AB = dx$ and $CD = dy$;

then $\qquad dz = Z - z = DP - DE = EP'$.

**178. Normal line to a surface.** The normal line to a surface at a given point is the line passing through the point perpendicular to the tangent plane to the surface at that point.

The direction cosines of any line perpendicular to the tangent plane (69) are proportional to

$$\frac{\partial F_1}{\partial x_1},\ \frac{\partial F_1}{\partial y_1},\ \frac{\partial F_1}{\partial z_1}.$$

(71) $$\therefore\ \frac{x - x_1}{\dfrac{\partial F_1}{\partial x_1}} = \frac{y - y_1}{\dfrac{\partial F_1}{\partial y_1}} = \frac{z - z_1}{\dfrac{\partial F_1}{\partial z_1}}$$

*are the equations of the normal line* to the surface $F(x, y, z) = 0$ at $(x_1, y_1, z_1)$.*

Similarly, from (70),

(72) $$\frac{x - x_1}{\dfrac{\partial z_1}{\partial x_1}} = \frac{y - y_1}{\dfrac{\partial z_1}{\partial y_1}} = \frac{z - z_1}{-1}$$

*are the equations of the normal line* to the surface $z = f(x, y)$ at $(x_1, y_1, z_1)$.*

---

* See footnote, p. 274.

## EXAMPLES

1. Find the equation of the tangent plane and the equations of the normal line to the sphere $x^2 + y^2 + z^2 = 14$ at the point (1, 2, 3).

*Solution.* Let $F(x, y, z) = x^2 + y^2 + z^2 - 14$;

then
$$\frac{\partial F}{\partial x} = 2x,\ \frac{\partial F}{\partial y} = 2y,\ \frac{\partial F}{\partial z} = 2z;\ x_1 = 1,\ y_1 = 2,\ z_1 = 3.$$

$$\therefore \frac{\partial F_1}{\partial x_1} = 2,\ \frac{\partial F_1}{\partial y_1} = 4,\ \frac{\partial F_1}{\partial z_1} = 6.$$

Substituting in (69), $2(x-1) + 4(y-2) + 6(z-3)$, or, $x + 2y + 3z = 14$, the tangent plane.

Substituting in (71),
$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{6},$$

giving $z = 3x$ and $2z = 3y$, equations of the normal line.

2. Find the equation of the tangent plane and the equations of the normal line to the ellipsoid $4x^2 + 9y^2 + 36z^2 = 36$, at point of contact where $x = 2$, $y = 1$, and $z$ is positive. *Ans.* Tangent plane, $8(x-2) + 9(y-1) + 6\sqrt{11}(z - \frac{1}{6}\sqrt{11}) = 0$,

normal line, $\dfrac{x-2}{8} = \dfrac{y-1}{9} = \dfrac{z - \frac{1}{6}\sqrt{11}}{6\sqrt{11}}$.

3. Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + 4y^2$ at the point (2, 1, 12). *Ans.* $8x + 8y - z = 12$.

4. Find the equations of the normal line to the hyperboloid of one sheet $x^2 - 4y^2 + 2z^2 = 6$ at (2, 2, 3). *Ans.* $y + 4x = 10$, $3x - z = 3$.

5. Find the equation of the tangent plane to the hyperboloid of two sheets $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ at $(x_1, y_1, z_1)$. *Ans.* $\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} - \dfrac{z_1z}{c^2} = 1$.

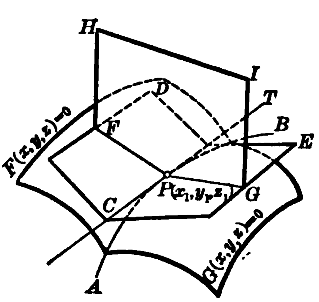
6. Find the equation of the tangent plane at the point $(x_1, y_1, z_1)$ on the surface $ax^2 + by^2 + cz^2 + d = 0$. *Ans.* $ax_1x + by_1y + cz_1z + d = 0$.

7. Show that the equation of the plane tangent to the sphere
$$x^2 + y^2 + z^2 + 2Lx + 2My + 2Nz + D = 0$$
at the point $(x_1, y_1, z_1)$ is
$$x_1x + y_1y + z_1z + L(x + x_1) + M(y + y_1) + N(z + z_1) + D = 0.$$

8. Find the equation of the tangent plane at any point of the surface
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}},$$
and show that the sum of the squares of the intercepts on the axes made by the tangent plane is constant.

9. Prove that the tetrahedron formed by the coördinate planes and any tangent plane to the surface $xyz = a^3$ is of constant volume.

**179. Another form of the equations of the tangent line to a skew curve.** If the curve in question be the curve of intersection $AB$ of the two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$, the tangent line $PT$ at $P(x_1, y_1, z_1)$ is the intersection of the tangent planes $CD$ and $CE$ at that point, for it is also tangent to both surfaces and hence must lie in both tangent planes. The equations of the two tangent planes at $P$ are, from (69),



$$(73)\qquad \begin{cases} \dfrac{\partial F_1}{\partial x_1}(x-x_1)+\dfrac{\partial F_1}{\partial y_1}(y-y_1)+\dfrac{\partial F_1}{\partial z_1}(z-z_1)=0,\\[2ex] \dfrac{\partial G_1}{\partial x_1}(x-x_1)+\dfrac{\partial G_1}{\partial y_1}(y-y_1)+\dfrac{\partial G_1}{\partial z_1}(z-z_1)=0. \end{cases}$$

Taken simultaneously, the equations (73) are *the equations of the tangent line $PT$ to the skew curve $AB$.* Equations (73) in more compact form are

$$(74)\qquad \frac{x-x_1}{\dfrac{\partial F_1}{\partial y_1}\dfrac{\partial G_1}{\partial z_1}-\dfrac{\partial F_1}{\partial z_1}\dfrac{\partial G_1}{\partial y_1}}=\frac{y-y_1}{\dfrac{\partial F_1}{\partial z_1}\dfrac{\partial G_1}{\partial x_1}-\dfrac{\partial F_1}{\partial x_1}\dfrac{\partial G_1}{\partial z_1}}=\frac{z-z_1}{\dfrac{\partial F_1}{\partial x_1}\dfrac{\partial G_1}{\partial y_1}-\dfrac{\partial F_1}{\partial y_1}\dfrac{\partial G_1}{\partial x_1}},$$

or, using determinants,

$$(75)\qquad \frac{x-x_1}{\begin{vmatrix}\dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial z_1}\\[2ex] \dfrac{\partial G_1}{\partial y_1} & \dfrac{\partial G_1}{\partial z_1}\end{vmatrix}}=\frac{y-y_1}{\begin{vmatrix}\dfrac{\partial F_1}{\partial z_1} & \dfrac{\partial F_1}{\partial x_1}\\[2ex] \dfrac{\partial G_1}{\partial z_1} & \dfrac{\partial G_1}{\partial x_1}\end{vmatrix}}=\frac{z-z_1}{\begin{vmatrix}\dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial y_1}\\[2ex] \dfrac{\partial G_1}{\partial x_1} & \dfrac{\partial G_1}{\partial y_1}\end{vmatrix}}.$$

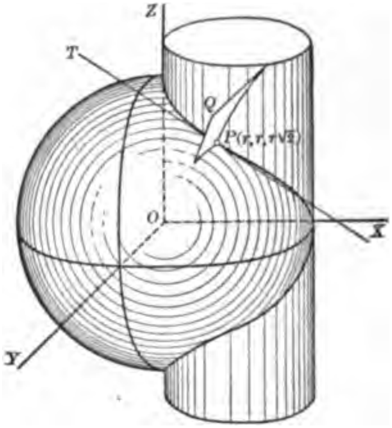
**180. Another form of the equation of the normal plane to a skew curve.** The *normal plane* to a skew curve at a given point has already been defined as the plane passing through that point perpendicular to the tangent line to the curve at that point. Thus, in the above figure, $PHI$ is the normal plane to the curve $AB$ at $P$. Since this plane is perpendicular to (75), we have at once,

$$(76)\qquad \begin{vmatrix}\dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial z_1}\\[2ex] \dfrac{\partial G_1}{\partial y_1} & \dfrac{\partial G_1}{\partial z_1}\end{vmatrix}(x-x_1)+\begin{vmatrix}\dfrac{\partial F_1}{\partial z_1} & \dfrac{\partial F_1}{\partial x_1}\\[2ex] \dfrac{\partial G_1}{\partial z_1} & \dfrac{\partial G_1}{\partial x_1}\end{vmatrix}(y-y_1)+\begin{vmatrix}\dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial y_1}\\[2ex] \dfrac{\partial G_1}{\partial x_1} & \dfrac{\partial G_1}{\partial y_1}\end{vmatrix}(z-z_1)=0,$$

*the equation of the normal plane to a skew curve.*

## EXAMPLES

**1. Find the equations of the tangent line and the equation of the normal plane at $(r, r, r\sqrt{2})$ to the curve of intersection of the sphere and cylinder whose equations are respectively $x^2 + y^2 + z^2 = r^2$, $x^2 + y^2 + 2rx$.**



*Solution.* Let $F = x^2 + y^2 + z^2 - r^2$ and $G = x^2 + y^2 - 2rx$.

$$\frac{\partial F_1}{\partial x_1} = 2r,\ \frac{\partial F_1}{\partial y_1} = 2r,\ \frac{\partial F_1}{\partial z_1} = 2\sqrt{2}r;$$

$$\frac{\partial G_1}{\partial x_1} = 0,\ \frac{\partial G_1}{\partial y_1} = 2r,\ \frac{\partial G_1}{\partial z_1} = 0.$$

Substituting in (75), $\dfrac{x - r}{-\sqrt{2}} = \dfrac{y - r}{0} = \dfrac{z - r\sqrt{2}}{1}$;

or, $$y = r,\ x + \sqrt{2}z = r - r\sqrt{2},$$

the equations of the tangent $PT$ at $P$ to the curve of intersection.

Substituting in (76), we get the equation of the normal plane,

$$-\sqrt{2}(x - r) + 0(y - r) + (z - r\sqrt{2}) = 0,$$

or, $$\sqrt{2}x - z = 0.$$

**2. Find the equations of the tangent line to the circle**

$$x^2 + y^2 + z^2 = 25,$$
$$x + z = 5,$$

at the point $(2, 2\sqrt{3}, 3)$. *Ans.* $2x + 2\sqrt{3}y + 3z = 25,\ x + z = 5$.

**3. Find the equation of normal plane to the curve**

$$x^2 + y^2 + z^2 = r^2,$$
$$x^2 - rx + y^2 = 0,$$

at $(x_1, y_1, z_1)$. *Ans.* $2y_1z_1x - (2x_1 - r)z_1y - ry_1z = 0$.

4. The equations of a helix (spiral) are

$$x^2 + y^2 = r^2,$$
$$y = x \tan \frac{z}{c}.$$

Show that at the point $(x_1, y_1, z_1)$ the equations of the tangent line are

$$c(x - x_1) + y_1(z - z_1) = 0,$$
$$c(y - y_1) - x_1(z - z_1) = 0;$$

and the equation of the normal plane is

$$y_1x - x_1y - c(z - z_1) = 0.$$

5. A skew curve is formed by the intersection of the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ and the sphere $x^2 + y^2 + z^2 = r^2$. Show that at the point $(x_1, y_1, z_1)$ the equations of the tangent line to the curve are

$$c^2(a^2 - b^2)x_1(x - x_1) = - a^2(b^2 + c^2)z_1(z - z_1),$$
$$c^2(a^2 - b^2)y_1(y - y_1) = + b^2(c^2 + a^2)z_1(z - z_1);$$

and the equation of the normal plane is

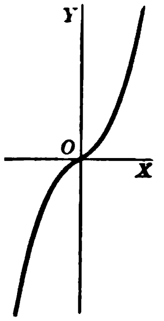
$$a^2(b^2 + c^2)y_1z_1x - b^2(c^2 + a^2)z_1x_1y - c^2(a^2 - b^2)x_1y_1z = 0.$$

# CHAPTER XXIII

## CURVES FOR REFERENCE

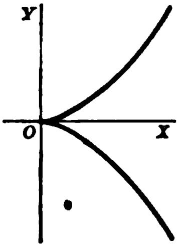
For the convenience of the student a number of the more common curves employed in the text are collected here.

### CUBICAL PARABOLA



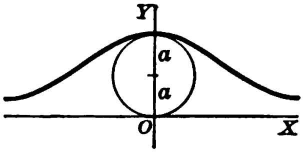
$y = ax^3$.

### SEMICUBICAL PARABOLA



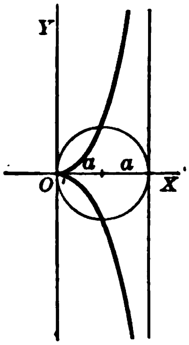
$y^2 = ax^3$.

### THE WITCH OF AGNESI



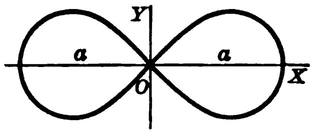
$x^2y = 4\,a^2(2\,a - y)$.

### THE CISSOID OF DIOCLES



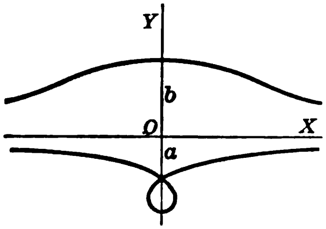
$y^2(2\,a - x) = x^3$.
$\rho = 2\,a \sin\theta \tan\theta$.

### THE LEMNISCATE OF BERNOULLI



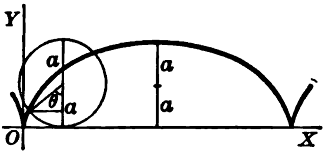
$$(x^2+y^2)^2=a^2(x^2-y^2).$$
$$\rho^2=a^2\cos 2\theta.$$

### THE CONCHOID OF NICOMEDES.



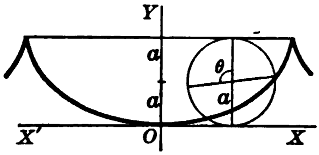
$$x^2y^2=(y+a)^2(b^2-y^2).$$
$$\rho=a\sec\theta\pm b.$$

### CYCLOID, ORDINARY CASE



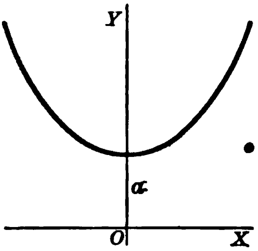
$$x=a \text{ arc vers } \frac{y}{a}-\sqrt{2ay-y^2}.$$
$$\begin{cases} x=a(\theta-\sin\theta), \\ y=a(1-\cos\theta). \end{cases}$$

### CYCLOID, VERTEX AT ORIGIN



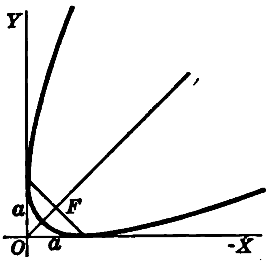
$$x=a \text{ arc vers } \frac{y}{a}+\sqrt{2ay-y^2}.$$
$$\begin{cases} x=a(\theta+\sin\theta), \\ y=a(1-\cos\theta). \end{cases}$$

### CATENARY



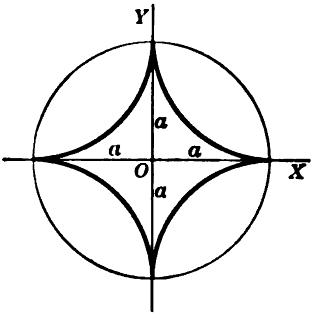
$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right).$$

### PARABOLA



$$x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}.$$

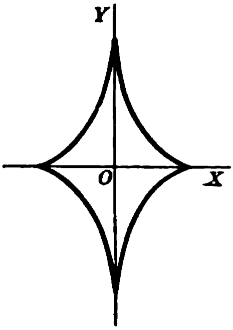
HYPOCYCLOID OF FOUR CUSPS



$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

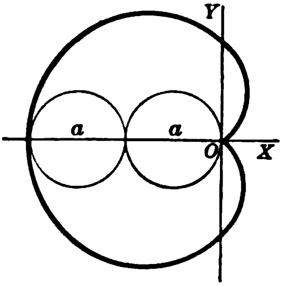
$$\begin{cases} x=a\cos^3\theta, \\ y=a\sin^3\theta. \end{cases}$$

EVOLUTE OF ELLIPSE



$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

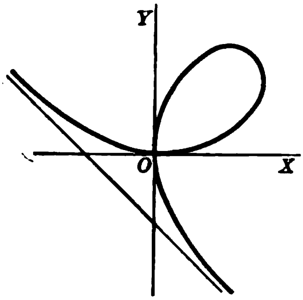
CARDIOID



$$x^2+y^2+ax=a\sqrt{x^2+y^2}.$$

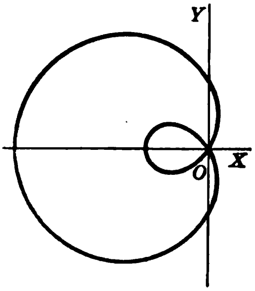
$$\rho=a(1-\cos\theta).$$

FOLIUM OF DESCARTES



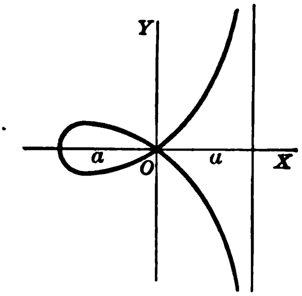
$$x^3+y^3-3axy=0.$$

### LIMAÇON



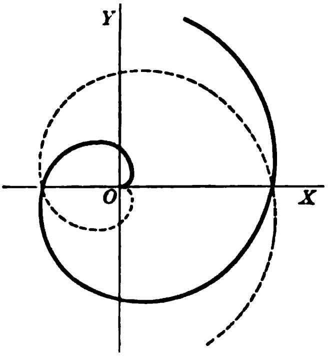
$$\rho = b - a\cos\theta.$$

### STROPHOID



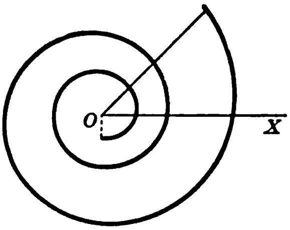
$$y^2 = x^2\,\frac{a+x}{a-x}.$$

### SPIRAL OF ARCHIMEDES



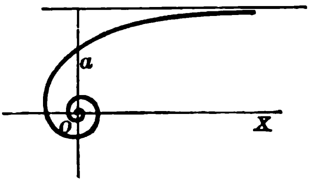
$$\rho = a\theta.$$

### LOGARITHMIC OR EQUIANGULAR SPIRAL



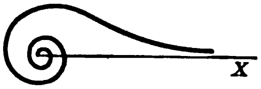
$$\rho = e^{a\theta}, \text{ or}$$
$$\log\rho = a\theta.$$

### HYPERBOLIC OR RECIPROCAL SPIRAL



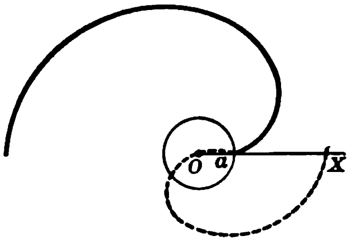
$$\rho\theta = a.$$

### LITUUS



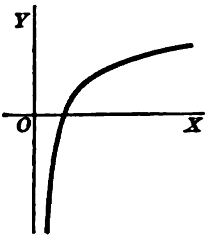
$$\rho^2\theta = a^2.$$

PARABOLIC SPIRAL



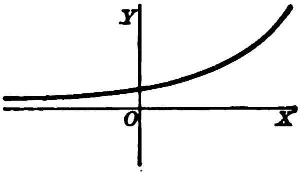
$(\rho - a)^2 = 4\,ac\,\theta.$

LOGARITHMIC CURVE



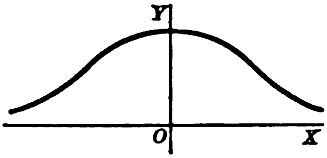
$y = \log x.$

EXPONENTIAL CURVE



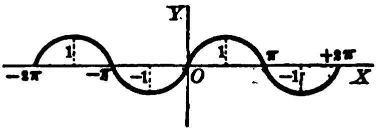
$y = e^x.$

PROBABILITY CURVE



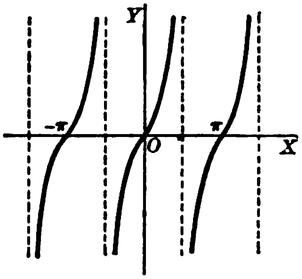
$y = e^{-x^2}.$

SINE CURVE



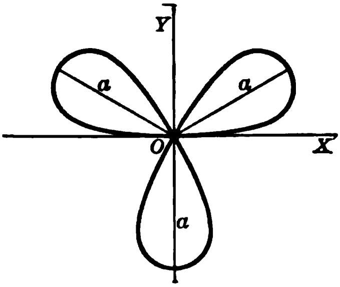
$y = \sin x.$

TANGENT CURVE



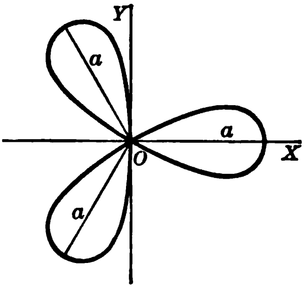
$y = \tan x.$

**THREE-LEAVED ROSE**



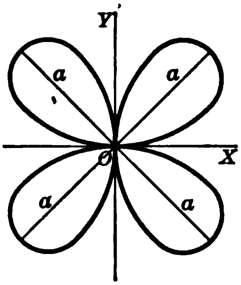
$\rho = a \sin 3\theta.$

**THREE-LEAVED ROSE**



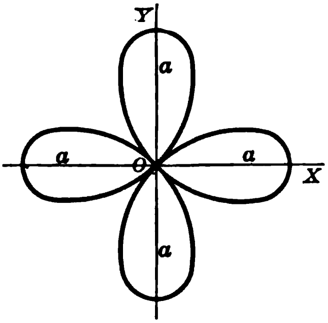
$\rho = a \cos 3\theta.$

**FOUR-LEAVED ROSE**



$\rho = a \sin 2\theta.$

**FOUR-LEAVED ROSE**



$\rho = a \cos 2\theta.$

# INTEGRAL CALCULUS

## CHAPTER XXIV

## INTEGRATION. RULES FOR INTEGRATING STANDARD ELEMENTARY FORMS

**181. Integration.** The student is already familiar with the mutually inverse operations of addition and subtraction, multiplication and division, involution and evolution. In the examples which follow the second members of one column are respectively the inverse of the second members of the other column.

$$\begin{aligned} y &= x^2 + 1, & x &= \pm \sqrt{y-1}; \\ y &= a^x, & x &= \log_a y; \\ y &= \sin x, & x &= \text{arc} \sin y. \end{aligned}$$

From the Differential Calculus we have learned how to calculate the derivative $f'(x)$ of a given function $f(x)$, an operation indicated by

$$\frac{d}{dx} f(x) = f'(x),$$

or, if we are using differentials, by

$$df(x) = f'(x)\,dx.$$

The problems of the Integral Calculus depend on the *inverse operation*, namely:

*To find a function $f(x)$ whose derivative*

$$(A) \qquad f'(x) = \phi(x)$$

*is given.*

Or, since it is customary to use differentials in the Integral Calculus, we may write

$$(B) \qquad df(x) = f'(x)\,dx = \phi(x)\,dx,$$

and state the problem as follows:

*Having given the differential of a function, find the function itself.*

The function $f(x)$ thus found is called an *integral*[*] of the given differential expression, the process of finding it is called *integration*, and the operation is indicated by writing the *integral sign*[†] $\int$ in front of the given differential expression. Thus,

$$(C)\qquad \int f'(x)\,dx\,{}^{\ddagger} = f(x),$$

read, *an integral of* $f'(x)\,dx$ *equals* $f(x)$. The differential $dx$ indicates that $x$ is *the variable of integration*. For example,

(a) If $f(x) = x^3$, then $f'(x)\,dx = 3\,x^2dx$, and

$$\int 3\,x^2dx = x^3.$$

(b) If $f(x) = \sin x$, then $f'(x)\,dx = \cos x dx$, and

$$\int \cos x dx = \sin x.$$

(c) If $f(x) = \text{arc}\tan x$, then $f'(x) = \frac{dx}{1+x^2}$, and

$$\int \frac{dx}{1+x^2} = \text{arc}\tan x.$$

Let us now emphasize what is apparent from the preceding explanations, namely, that

*Differentiation and integration are inverse operations.*

Differentiating $(C)$ gives

$$(D)\qquad d\int f'(x)\,dx = f'(x)\,dx.$$

Substituting the value of $f'(x)\,dx\,[= df(x)]$ from $(B)$ in $(C)$, we get

$$(E)\qquad \int df(x) = f(x).$$

Therefore, considered as symbols of operation, $\frac{d}{dx}$ and $\int \cdots dx$ are *inverse to each other*; or, if we are using differentials, $d$ and $\int$ are inverse to each other.

---

[*] Called *anti-differential* by some writers.

[†] Historically this sign is a distorted $S$, the initial letter of the word *sum*. Instead of defining integration as the inverse of differentiation we may define it as a process of summation, a very important notion which we will consider in Chapter XXX.

[‡] Some authors write this $D_x^{-1}f'(x)$ when they wish to emphasize the fact that it is an inverse operation.

When $d$ is followed by $\int$ they annul each other, as in ($D$), but when $\int$ is followed by $d$, as in ($E$), that will not in general be the case unless we ignore the *constant of integration.* The reason for this will appear at once from the definition of the constant of integration given in the next section.

**182. Constant of integration. Indefinite integral.** From the preceding section it follows that

$$\text{since } d(x^3) \quad = 3x^2dx, \text{ we have } \int 3x^2dx = x^3;$$

$$\text{since } d(x^3+2) = 3x^2dx, \text{ we have } \int 3x^2dx = x^3+2;$$

$$\text{since } d(x^3-7) = 3x^2dx, \text{ we have } \int 3x^2dx = x^3-7.$$

In fact, since

$$d(x^3+C) = 3x^2dx,$$

where $C$ is any arbitrary constant, we have

$$\int 3x^2dx = x^3 + C.$$

A constant $C$ arising in this way is called a *constant of integration.*[*] Since we can give $C$ as many values as we please, it follows that if a given differential expression has one integral, it has infinitely many differing only by constants. Hence

$$\int f'(x)\,dx = f(x) + C;$$

and since $C$ is unknown and *indefinite*, the expression

$$f(x) + C$$

is called the *indefinite integral of* $f'(x)\,dx$.

It is evident that if $\phi(x)$ is a function the derivative of which is $f(x)$, then $\phi(x) + C$, where $C$ is any constant whatever, is likewise a function the derivative of which is $f(x)$. Hence the

**Theorem.** *If two functions differ by a constant, they have the same derivative.*

---

[*] Constant here means that it is independent of the *variable of integration.*

It is, however, not obvious that if $\phi(x)$ is a function the derivative of which is $f(x)$, then *all* functions having the same derivative $f(x)$ are of the form

$$\phi(x)+C,$$

where $C$ is any constant. In other words, there remains to be proven the

**Converse Theorem.** *If two functions have the same derivative, their difference is a constant.*

*Proof.* Let $\phi(x)$ and $\psi(x)$ be two functions having the common derivative $f(x)$. Place

$$F(x)=\phi(x)-\psi(x);\text{ then}$$

$$(A)\quad F'(x)=\frac{d}{dx}[\phi(x)-\psi(x)]=f(x)-f(x)=0.\quad\text{By hypothesis}$$

But from the Theorem of Mean Value, (44), p. 168, we have

$$F(x+\Delta x)-F(x)=\Delta x F'(x+\theta\cdot\Delta x).\qquad 0<\theta<1$$

$$\therefore F(x+\Delta x)-F(x)=0,$$

[Since by $(A)$ the derivative of $F(x)$ is zero for all values of $x$.]

and
$$F(x+\Delta x)=F(x).$$

This means that the function

$$F(x)=\phi(x)-\psi(x)$$

does not change in value at all when $x$ takes on the increment $\Delta x$, i.e. $\phi(x)$ and $\psi(x)$ differ only by a constant.

In any given case the value of $C$ can be found when we know the value of the integral for some value of the variable, and this will be illustrated by numerous examples in the next chapter. For the present we shall content ourselves with first learning how to find the indefinite integrals of given differential expressions. In what follows we shall assume that *every continuous function has an indefinite integral*, a statement the rigorous proof of which is beyond the scope of this book. For all elementary functions, however, the truth of the statement will appear in the chapters which follow.

In all cases of indefinite integration the test to be applied in verifying the results is that *the differential of the integral must be equal to the given differential expression.*

**183. Rules for integrating standard elementary forms.** The Differential Calculus furnished us with a General Rule for differentiation (p. 42). The Integral Calculus gives us no corresponding general rule that can be readily applied in practice for performing the inverse operation of integration.[*] Each case requires special treatment and we arrive at the integral of a given differential expression through our previous knowledge of the known results of differentiation. That is, we must be able to answer the question,

*What function, when differentiated, will yield the given differential expression?*

Integration then is essentially a tentative process, and to expedite the work, tables of known integrals are formed called *standard forms*. To effect any integration we compare the given differential expression with these forms, and if it is found to be identical with one of them, the integral is known. If it is not identical with one of them, we strive to reduce it to one of the standard forms by various methods, many of which employ artifices which can be suggested by practice only. Accordingly a large portion of our treatise on the Integral Calculus will be devoted to the explanation of methods for integrating those functions which frequently appear in the process of solving practical problems.

From any result of differentiation may always be derived a formula for integration.

The following two rules are useful in reducing differential expressions to standard forms.

(*a*) *The integral of any algebraic sum of differential expressions equals the same algebraic sum of the integrals of these expressions taken separately.*

*Proof.* Differentiating the expression

$$\int du + \int dv - \int dw,$$

$u$, $v$, $w$ being functions of a single variable, we get

$$du + dv - dw. \qquad \text{III, p. 144}$$

$$[1] \qquad \therefore \int (du + dv - dw) = \int du + \int dv - \int dw.$$

---

[*] Even though the integral of a given differential expression may be known to exist, yet it may not be possible for us actually to find it in terms of known functions, because there are functions other than the elementary functions whose derivatives are elementary functions.

(*b*) *A constant factor may be written either before or after the integral sign.*

*Proof.* Differentiating the expression

$$a\int dv$$

gives $$adv.$$ IV, p. 144

[2] $$\therefore \int adv = a\int dv.$$

On account of their importance we shall write the above two rules as formulas at the head of the following list of

### STANDARD ELEMENTARY FORMS

[1] $$\int (du + dv - dw) = \int du + \int dv - \int dw.$$

[2] $$\int adv = a\int dv.$$

[3] $$\int dx = x + C.$$

[4] $$\int v^n dv = \frac{v^{n+1}}{n+1} + C. \qquad n \neq -1$$

[5] $$\int \frac{dv}{v} = \log v + C$$
$$= \log v + \log c = \log cv.$$
[Placing $C = \log c$.]

[6] $$\int a^v dv = \frac{a^v}{\log a} + C.$$

[7] $$\int e^v dv = e^v + C.$$

[8] $$\int \sin v\, dv = -\cos v + C.$$

[9] $$\int \cos v\, dv = \sin v + C.$$

[10] $$\int \sec^2 v\, dv = \tan v + C.$$

[11] $$\int \operatorname{cosec}^2 v\, dv = -\cot v + C.$$

[12] $$\int \sec v \tan v\, dv = \sec v + C.$$

[13] $$\int \csc v \cot v\, dv = -\csc v + C.$$

[14] $$\int \tan v\, dv = \log \sec v + C.$$

[15] $$\int \cot v\, dv = \log \sin v + C.$$

[16] $$\int \sec v\, dv = \log \tan\left(\frac{v}{2} + \frac{\pi}{4}\right) + C.$$

[17] $$\int \csc v\, dv = \log \tan\left(\frac{v}{2}\right) + C.$$

[18] $$\int \frac{dv}{v^2 + a^2} = \frac{1}{a} \operatorname{arc\,tan} \frac{v}{a} + C.$$

[19] $$\int \frac{dv}{v^2 - a^2} = \frac{1}{2a} \log \frac{v-a}{v+a} + C.$$

[20] $$\int \frac{dv}{\sqrt{a^2 - v^2}} = \operatorname{arc\,sin} \frac{v}{a} + C.$$

[21] $$\int \frac{dv}{\sqrt{v^2 \pm a^2}} = \log(v + \sqrt{v^2 \pm a^2}) + C.$$

[22] $$\int \frac{dv}{\sqrt{2av - v^2}} = \operatorname{arc\,vers} \frac{v}{a} + C.$$

[23] $$\int \frac{dv}{v\sqrt{v^2 - a^2}} = \frac{1}{a} \operatorname{arc\,sec} \frac{v}{a} + C.$$

*Proof of* [3]. Since

$$d(x + C) = dx, \qquad \text{II, p. 144}$$

we get

$$\int dx = x + C.$$

*Proof of* [4]. Since

$$d\left(\frac{v^{n+1}}{n+1} + C\right) = v^n dv, \qquad \text{VII, p. 144}$$

we get

$$\int v^n dv = \frac{v^{n+1}}{n+1} + C.$$

This holds true for all values of $n$ except $n = -1$. For, when $n = -1$, [4] gives

$$\int v^{-1} dv = \frac{v^{-1+1}}{-1+1} + C = \frac{1}{0} + C = \infty + C,$$

which has no meaning.

The case when $n = -1$ comes under [5].

*Proof of* [5]. Since

$$d(\log v + C) = \frac{dv}{v}, \qquad \text{IX, p. 145}$$

we get

$$\int \frac{dv}{v} = \log v + C.$$

The results we get from [5] may be put in more compact form if we denote the constant of integration by $\log c$. Thus,

$$\int \frac{dv}{v} = \log v + \log c = \log cv.$$

Formula [5] states that *if the expression under the integral sign is a fraction whose numerator is the differential of the denominator, then the integral is the natural logarithm of the denominator.*

## EXAMPLES

For formulas [1]-[5].

Verify the following integrations.

1. $\int x^6 dx = \frac{x^7}{7} + C$, by [4], where $v = x$ and $n = 6$.

2. $$\int ax^3 dx = a\int x^3 dx \qquad \text{by [2]}$$
$$= \frac{ax^4}{4} + C. \qquad \text{By [4]}$$

3. $$\int (2x^3 - 5x^2 - 3x + 4)dx = \int 2x^3 dx - \int 5x^2 dx - \int 3x dx + \int 4 dx \qquad \text{by [1]}$$
$$= 2\int x^3 dx - 5\int x^2 dx - 3\int x dx + 4\int dx \qquad \text{by [2]}$$
$$= \frac{x^4}{2} - \frac{5x^3}{3} - \frac{3x^2}{2} + 4x + C.$$

*Note.* Although each separate integration requires an arbitrary constant, we write down only a single constant denoting their algebraic sum.

4. $$\int \left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)dx = \int 2ax^{-\frac{1}{2}}dx - \int bx^{-2}dx + \int 3cx^{\frac{2}{3}}dx \qquad \text{by [1]}$$
$$= 2a\int x^{-\frac{1}{2}}dx - b\int x^{-2}dx + 3c\int x^{\frac{2}{3}}dx \qquad \text{by [2]}$$
$$= 2a \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - b \cdot \frac{x^{-1}}{-1} + \frac{3c \cdot x^{\frac{5}{3}}}{\frac{5}{3}} + C \qquad \text{by [4]}$$
$$= 4a\sqrt{x} + \frac{b}{x} + \tfrac{9}{5}cx^{\frac{5}{3}} + C.$$

5. $\int 2ax^{b-1}dx = \frac{2ax^b}{b} + C.$

6. $\int 3mz^6 dz = \frac{3mz^7}{7} + C.$

7. $\int\left(bs^3+\frac{1}{s^{\frac{3}{2}}}\right)ds=\frac{bs^4}{4}-\frac{2}{\sqrt{s}}+C.$

8. $\int\sqrt{2\,px}\,dx=\frac{2}{3}\,x\sqrt{2\,px}+C.$

9. $\int(a^{\frac{2}{3}}-x^{\frac{2}{3}})^3\,dx=a^2x-\frac{9}{7}\,a^{\frac{2}{3}}x^{\frac{7}{3}}+\frac{9}{5}\,a^{\frac{4}{3}}x^{\frac{5}{3}}-\frac{x^3}{3}+C.$

***Hint.*** **First expand.**

10. $\int(a^2-y^2)^3\sqrt{y}\,dy=2\,y^{\frac{3}{2}}\left(\frac{a^6}{3}-\frac{3\,a^4y^2}{7}+\frac{3\,a^2y^4}{11}-\frac{y^6}{15}\right)+C.$

11. $\int(\sqrt{a}-\sqrt{t})^3dt=a^{\frac{3}{2}}t-2\,at^{\frac{3}{2}}+\frac{3\,a^{\frac{1}{2}}t^2}{2}-\frac{2\,t^{\frac{5}{2}}}{5}+C.$

12. $\int\frac{dx}{(nx)^{\frac{n-1}{n}}}=(nx)^{\frac{1}{n}}+C.$

13. $\int(a^2+b^2x^2)^{\frac{1}{2}}\,xdx=\frac{(a^2+b^2x^2)^{\frac{3}{2}}}{3\,b^2}+C.$

***Hint.*** **This may be brought to form [4]. For let $v=a^2+b^2x^2$ and $n=\frac{1}{2}$; then $dv=2\,b^2xdx$. If we now insert the constant factor $2\,b^2$ before $xdx$, and its reciprocal $\frac{1}{2\,b^2}$ before the integral sign (so as not to change the value of the expression), the expression may be integrated, using [4], namely,**

$$\int v^ndv=\frac{v^{n+1}}{n+1}+C.$$

**Thus,** $$\int(a^2+b^2x^2)^{\frac{1}{2}}xdx=\frac{1}{2\,b^2}\int(a^2+b^2x^2)^{\frac{1}{2}}\,2\,b^2xdx=\frac{1}{2\,b^2}\int(a^2+b^2x^2)^{\frac{1}{2}}d(a^2+b^2x^2)$$

$$=\frac{1}{2\,b^2}\cdot\frac{(a^2+b^2x^2)^{\frac{3}{2}}}{\frac{3}{2}}+C=\frac{(a^2+b^2x^2)^{\frac{3}{2}}}{3\,b^2}+C.$$

***Note.*** **The student is warned against transferring any function of the variable from one side of the integral sign to the other, since that would change the value of the integral.**

14. $\int\sqrt{a^2-x^2}\,xdx=-\frac{1}{3}\,(a^2-x^2)^{\frac{3}{2}}+C.$

15. $\int(3\,ax^2+4\,bx^3)^{\frac{1}{3}}\,(2\,ax+4\,bx^2)\,dx=\frac{1}{4}\,(3\,ax^2+4\,bx^3)^{\frac{4}{3}}+C.$

***Hint.*** **Use [4], making $v=3\,ax^2+4\,bx^3$ and $n=\frac{1}{3}$.**

16. $\int b\,(6\,ax^2+8\,bx^3)^{\frac{1}{3}}\,(2\,ax+4\,bx^2)\,dx=\frac{b}{16}\,(6\,ax^2+8\,bx^3)^{\frac{4}{3}}+C.$

17. $\int\frac{x^2dx}{(a^2+x^3)^{\frac{1}{2}}}=\frac{2}{3}\,(a^2+x^3)^{\frac{1}{2}}+C.$

***Hint.*** **Write this $\int(a^2+x^3)^{-\frac{1}{2}}\,x^2dx$ and apply [4].**

18. $\int\frac{dx}{\sqrt{1-x}}=-\,2\sqrt{1-x}+C.$

19. $\int 2\,\pi y\left(\frac{y^2}{p^2}+1\right)^{\frac{1}{2}}dy=\frac{2\,\pi}{3\,p}\,(y^2+p^2)^{\frac{3}{2}}+C.$

20. $\displaystyle\int\frac{2\,as\,ds}{(b^2-c^2s^2)^2}=\frac{a}{c^2(b^2-c^2s^2)}+C.$

21. $\displaystyle\int\frac{3\,ax\,dx}{b^2+e^2x^2}=\frac{3\,a}{2\,e^2}\log(b^2+e^2x^2)+C.$

*Solution.* $\displaystyle\int\frac{3\,ax\,dx}{b^2+e^2x^2}=3\,a\int\frac{x\,dx}{b^2+e^2x^2}+C.$ **By** [2

This resembles [5]. For let $v=b^2+e^2x^2$; then $dv=2e^2x\,dx$. If we introduce the factor $2e^2$ after the integral sign, and $\frac{1}{2e^2}$ before it, we have not changed the value of the expression, but the numerator is now seen to be the differential of the denominator. Therefore

$$3\,a\int\frac{x\,dx}{b^2+e^2x^2}=\frac{3\,a}{2\,e^2}\int\frac{2\,e^2x\,dx}{b^2+e^2x^2}=\frac{3\,a}{2\,e^2}\int\frac{d(b^2+e^2x^2)}{b^2+e^2x^2}=\frac{3\,a}{2\,e^2}\log(b^2+e^2x^2)+C.$$ **By** [5

22. $\displaystyle\int\frac{x\,dx}{x^2-1}=\tfrac{1}{2}\log(x^2-1)+\log C=\log c\sqrt{x^2-1}.$

23. $\displaystyle\int\frac{(x^2-a^2)\,dx}{x^3-3\,a^2x}=\log c\,(x^3-3\,a^2x)^{\frac{1}{3}}.$

24. $\displaystyle\int\frac{5\,x^2dx}{10\,x^3+15}=\log c\,(10\,x^3+15)^{\frac{1}{6}}.$

25. $\displaystyle\int\frac{5\,bx\,dx}{8\,a-6\,bx^2}=\log\frac{c}{(8\,a-6\,bx^2)^{\frac{5}{12}}}.$

26. $\displaystyle\int\frac{x^3dx}{x+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\log(x+1)+C.$

*Hint.* First divide the numerator by the denominator.

27. $\displaystyle\int\frac{2\,x-1}{2\,x+3}dx=x-\log(2\,x+3)^2+C.$

28. $\displaystyle\int\frac{x^{n-1}-1}{x^n-nx}dx=\frac{1}{n}\log(x^n-nx)+C.$

29. $\displaystyle\int\frac{(y^2-2)^3\,dy}{y^5}=\frac{2}{y^4}-\frac{6}{y^2}+\frac{y^2}{2}-\log y^6+C.$

30. $\displaystyle\int\frac{t^{n-1}dt}{a+bt^n}=\frac{1}{nb}\log(a+bt^n)+C.$

31. $\displaystyle\int(\log a)^3\frac{da}{a}=\tfrac{1}{4}(\log a)^4+C.$

32. $\displaystyle\int\frac{r^2+1}{r-1}dr=\frac{r^2}{2}+r+2\log(r-1)+C.$

33. $\displaystyle\int\frac{u^{n-1}\,du}{(a+bu^n)^m}=\frac{(a+bu^n)^{1-m}}{bn(1-m)}+C.$

*Proofs of* [6] *and* [7]. These follow at once from the corresponding formula for differentiation, X and X *a*, p. 145.

### EXAMPLES

**For formulas [6] and [7].**

**Verify the following integrations.**

1. $\int ba^{2x}dx = \frac{ba^{2x}}{2\log a} + C.$

*Solution.* $\int ba^{2x}dx = b\int a^{2x}dx.$ By [2]

This resembles [6]. Let $v=2x$; then $dv=2dx$. If we then insert the factor 2 before $dx$ and the factor $\frac{1}{2}$ before the integral sign, we have

$$b\int a^{2x}dx = \frac{b}{2}\int a^{2x}2\,dx = \frac{b}{2}\int a^{2x}d(2x) = \frac{b}{2}\cdot\frac{a^{2x}}{\log a} + C. \qquad \text{By [6]}$$

2. $\int 3e^{x}dx = 3e^{x} + C.$

3. $\int (e^{5x} + a^{5x})\,dx = \frac{1}{5}\left(e^{5x} + \frac{a^{5x}}{\log a}\right) + C.$

4. $\int e^{\frac{x}{n}}dx = ne^{\frac{x}{n}} + C.$

5. $\int e^{x^2+4x+3}(x+2) = \frac{1}{2}e^{x^2+4x+3} + C.$

6. $\int (a^{nx} - b^{mx})\,dx = \frac{a^{nx}}{n\log a} - \frac{b^{mx}}{m\log b} + C.$

7. $\int a^{x}e^{x}dx = \frac{a^{x}e^{x}}{1+\log a} + C.$

8. $\int (e^{\frac{x}{a}} + e^{-\frac{x}{a}})\,dx = a(e^{\frac{x}{a}} - e^{-\frac{x}{a}}) + C.$

9. $\int (e^{y} + e^{-y})^2\,dy = \frac{1}{2}(e^{2y} - e^{-2y}) + 2y + C.$

10. $\int (3e^{2t} - 1)^{\frac{1}{2}}e^{2t}dt = \frac{1}{9}(3e^{2t} - 1)^{\frac{3}{2}} + C.$

11. $\int \frac{e^{3s}ds}{e^{s}-1} = \frac{e^{2s}}{2} + e^{s} + \log(e^{s}+1) + C.$

12. $\int \frac{e^{r}-1}{e^{r}+1}dr = \log(e^{r}+1)^2 - r + C.$

13. $\int \frac{(a^{x}-b^{x})^2}{a^{x}b^{x}}dx = \frac{a^{x}b^{-x} - a^{-x}b^{x}}{\log a - \log b} - 2x + C.$

*Proofs of* [8]–[13]. These follow at once from the corresponding formulas for differentiation, XII, etc., p. 145.

*Proof of* [14]. $$\int \tan v dv = \int \frac{\sin v dv}{\cos v} = -\int \frac{-\sin v dv}{\cos v}$$
$$= -\int \frac{d(\cos v)}{\cos v}$$
$$= -\log \cos v + C \qquad \text{by [5]}$$
$$= \log \sec v + C.$$

[Since $-\log \cos v = -\log \frac{1}{\sec v} = -\log 1 + \log \sec v = \log \sec v$.]

*Proof of* [15]. $$\int \cot v dv = \int \frac{\cos v dv}{\sin v} = \int \frac{d(\sin v)}{\sin v}$$
$$= \log \sin v + C. \qquad \text{By [5]}$$

*Proof of* [17]. Since $\csc v = \csc v \dfrac{\csc v - \cot v}{\csc v - \cot v}$,

$$\int \csc v dv = \int \frac{-\csc v \cot v + \csc^2 v}{\csc v - \cot v} dv$$
$$= \int \frac{d(\csc v - \cot v)}{\csc v - \cot v}$$
$$= \log(\csc v - \cot v) + C \qquad \text{by [5]}$$
$$= \log\left(\frac{1 - \cos v}{\sin v}\right) + C$$
$$= \log \frac{2 \sin^2 \frac{v}{2}}{2 \sin \frac{v}{2} \cos \frac{v}{2}} + C \quad \text{by 37, p. 2; 39, p. 3}$$
$$= \log \tan \frac{v}{2} + C.$$

*Proof of* [16]. Substituting $v + \frac{\pi}{2}$ for $v$ in [17] gives

$$\int \csc\left(v + \frac{\pi}{2}\right) dv = \log \tan\left(\frac{v}{2} + \frac{\pi}{4}\right) + C.$$

But $\csc\left(v + \frac{\pi}{2}\right) = \sec v$; therefore

$$\int \sec v dv = \log \tan\left(\frac{v}{2} + \frac{\pi}{4}\right) + C.$$

## EXAMPLES

**For formulas [8]–[17].**

**Verify the following integrations.**

1. $\int \sin 2\,ax dx = -\frac{\cos 2\,ax}{2\,a} + C.$

*Solution.* This resembles [8]. For let $v = 2\,ax$; then $dv = 2\,adx$. If we now insert the factor $2\,a$ before $dx$ and the factor $\frac{1}{2\,a}$ before the integral sign, we get

$$\int \sin 2\,ax dx = \frac{1}{2\,a}\int \sin 2\,ax \cdot 2\,adx$$

$$= \frac{1}{2\,a}\int \sin 2\,ax \cdot d(2\,ax) = \frac{1}{2\,a} \cdot -\cos 2\,ax + C \qquad \text{by [8]}$$

$$= -\frac{\cos 2\,ax}{2\,a} + C.$$

2. $\int \cos mx dx = \frac{1}{m}\sin mx + C.$

3. $\int 5 \sec^2 bx dx = \frac{5}{b}\tan bx + C.$

4. $\int \left(\cos\frac{\theta}{3} - \sin 3\,\theta\right)d\theta = 3\sin\frac{\theta}{3} + \frac{1}{3}\cos 3\,\theta + C.$

5. $\int 7 \sec 3\,\alpha \tan 3\,\alpha d\alpha = \frac{7}{3}\sec 3\,\alpha + C.$

6. $\int k \cos(a + by)\,dy = \frac{k}{b}\sin(a + by) + C.$

7. $\int \operatorname{cosec}^2 x^3 \cdot x^2 dx = -\frac{1}{3}\cot x^3 + C.$

8. $\int 4 \csc ax \cot ax dx = -\frac{4}{a}\csc ax + C.$

9. $\int \frac{\sin x dx}{a + b\cos x} = \log\frac{c}{(a + b\cos x)^{\frac{1}{b}}}.$

10. $\int e^{\cos x}\sin x dx = -e^{\cos x} + C.$

11. $\int \frac{dx}{\cos^2(a - bx)} = -\frac{1}{b}\tan(a - bx) + C.$

12. $\int \cos(\log x)\frac{dx}{x} = \sin\log x + C.$

14. $\int \frac{(1 + \cos x)\,dx}{x + \sin x} = \log(x + \sin x)\,c.$

13. $\int \frac{dx}{\sin^2\frac{x}{n}} = -n\cot\frac{x}{n} + C.$

15. $\int \frac{\sin\phi d\phi}{\cos^2\phi} = \sec\phi + C.$

16. $\int (\tan\alpha + \cot\alpha)^2 d\alpha = \tan\alpha - \cot\alpha + C.$

17. $\int (\sec\beta - \tan\beta)^2 d\beta = 2(\tan\beta - \sec\beta) - \beta + C.$

18. $\int a^{1+\sin\theta}\cos\theta\, d\theta = \frac{a^{1+\sin\theta}}{\log a} + C.$

19. $\int (\tan 2u - 1)^2 du = \frac{1}{2}\tan 2u + \log\cos 2u + C.$

20. $\int \tan y \tan(y + a)\, dy = -y - \frac{\log(1 - \tan a \tan y)}{\tan a}.$

*Proof of* [18]. Since

$$d\left(\text{arc}\tan\frac{v}{a} + C\right) = \frac{d\left(\frac{v}{a}\right)}{1 + \left(\frac{v}{a}\right)^2} = \frac{dv}{v^2 + a^2}, \qquad \text{by XIV, p. 145}$$

we get
$$\int \frac{dv}{v^2 + a^2} = \text{arc}\tan\frac{v}{a} + C.^*$$

*Proof of* [19]. Since $\frac{1}{v^2 - a^2} = \frac{1}{2a}\left(\frac{1}{v - a} - \frac{1}{v + a}\right)$,

$$\begin{aligned}\int \frac{dv}{v^2 - a^2} &= \frac{1}{2a}\int\left(\frac{1}{v - a} - \frac{1}{v + a}\right)dv \\ &= \frac{1}{2a}\{\log(v - a) - \log(v + a)\} + C \qquad \text{by [5]} \\ &= \frac{1}{2a}\log\frac{v - a}{v + a} + C.\end{aligned}$$

*Proof of* [20]. Since

$$d\left(\text{arc}\sin\frac{v}{a} + C\right) = \frac{d\left(\frac{v}{a}\right)}{\sqrt{1 - \left(\frac{v}{a}\right)^2}} = \frac{dv}{\sqrt{a^2 - v^2}}, \qquad \text{by XIX, p. 145}$$

we get
$$\int \frac{dv}{\sqrt{a^2 - v^2}} = \text{arc}\sin\frac{v}{a} + C.$$

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* Also $d\left(\text{arc}\cot\frac{v}{a} + C\right) = -\frac{dv}{v^2 + a^2}$ and $\int \frac{dv}{v^2 + a^2} = -\text{arc}\cot\frac{v}{a} + C.'$ Hence

$$\int \frac{dv}{v^2 + a^2} = \text{arc}\tan\frac{v}{a} + C = -\text{arc}\cot\frac{v}{a} + C.'$$

Since $\text{arc}\tan\frac{v}{a} + \text{arc}\cot\frac{v}{a} = \frac{\pi}{2}$, we see that one result may be easily transformed into the other. The same kind of discussion may be given for [20] involving $\text{arc}\sin\frac{v}{a}$ and $\text{arc}\cos\frac{v}{a}$, and for [23] involving $\text{arc}\sec\frac{v}{a}$ and $\text{arc}\csc\frac{v}{a}$.

***Proof of*** **[21].** Assume $\sqrt{v^2 + a^2} = z$, a new variable.

Then $v^2 + a^2 = z^2$, and differentiating, $2\,v dv = 2\,z dz$, or,

$$\frac{dv}{z} = \frac{dz}{v}.$$

By composition in proportion,

$$\frac{dv}{z} = \frac{dz}{v} = \frac{dv + dz}{v + z};$$

therefore
$$\frac{dv}{\sqrt{v^2 + a^2}} = \frac{dv + dz}{v + z}.$$

[Replacing $z$ by its equal $\sqrt{v^2 + a^2}$.]

Hence
$$\int \frac{dv}{\sqrt{v^2 + a^2}} = \int \frac{dv + dz}{v + z}$$
$$= \log (v + z) + C \qquad \text{by [5]}$$
$$= \log (v + \sqrt{v^2 + a^2}) + C.$$

In the same way by assuming

$$\sqrt{v^2 - a^2} = z,$$

we get
$$\int \frac{dv}{\sqrt{v^2 - a^2}} = \log (v + \sqrt{v^2 - a^2}) + C.$$

*Proofs of* [22] *and* [23]. These follow at once from the corresponding formulas for differentiation, XXIII and XXV, p. 48.

### EXAMPLES

For formulas [18]–[23].

Verify the following integrations.

1. $\int \frac{dx}{4x^2 + 9} = \frac{1}{6} \arctan \frac{2x}{3} + C.$

*Solution.* This resembles [18]. For, let $v^2 = 4x^2$ and $a^2 = 9$; then $v = 2x$, $dv = 2\,dx$, and $a = 3$. Hence if we multiply the numerator by 2 and divide in front of the integral sign by 2, we get

$$\int \frac{dx}{4x^2 + 9} = \frac{1}{2} \int \frac{2\,dx}{(2x)^2 + (3)^2} = \frac{1}{2} \int \frac{d(2x)}{(2x)^2 + (3)^2}$$
$$= \frac{1}{6} \arctan \frac{2x}{3} + C. \qquad \text{By [18]}$$

2. $\int \frac{dx}{9x^2 - 4} = \frac{1}{12} \log \frac{3x - 2}{3x + 2} + C.$ 3. $\int \frac{dx}{\sqrt{16 - 9x^2}} = \frac{1}{3} \arcsin \frac{3x}{4} + C.$

4. $$\int\frac{dx}{\sqrt{b^2+e^2x^2}}=\frac{1}{e}\log\left(ex+\sqrt{b^2+e^2x^2}\right)+C.$$

5. $$\int\frac{dy}{\sqrt{b^2y^2-a^2}}=\frac{1}{b}\log\left(by+\sqrt{b^2y^2-a^2}\right)+C.$$

6. $$\int\frac{5\,x\,dx}{\sqrt{1-x^4}}=\frac{5}{2}\,\text{arc sin}\,x^2+C.$$

8. $$\int\frac{dx}{x\sqrt{4\,x^2-9}}=\frac{1}{3}\,\text{arc sec}\,\frac{2\,x}{3}+C.$$

7. $$\int\frac{ax\,dx}{x^4+e^4}=\frac{a}{2\,e^2}\,\text{arc tan}\,\frac{x^2}{e^2}+C.$$

9. $$\int\frac{dx}{\sqrt{6\,x-x^2}}=\text{arc vers}\,\frac{x}{3}+C.$$

10. $$\int\frac{e\,dt}{a^2-b^2t^2}=\frac{e}{2\,ab}\log\frac{bt+a}{bt-a}+C.$$

11. $$\int\frac{7\,ds}{\sqrt{3-5\,s^2}}=\frac{7}{\sqrt{5}}\,\text{arc sin}\sqrt{\frac{5}{3}}\,s+C.$$

12. $$\int\frac{dv}{\sqrt{av^2-b}}=\frac{1}{\sqrt{a}}\log\left(\sqrt{a}\,v+\sqrt{av^2-b}\right)+C.$$

13. $$\int\frac{\cos\alpha\,d\alpha}{a^2+\sin^2\alpha}=\frac{1}{a}\,\text{arc tan}\left(\frac{\sin\alpha}{a}\right)+C.$$

14. $$\int\frac{e^t\,dx}{\sqrt{1-e^{2t}}}=\text{arc sin}\,e^t+C.$$

15. $$\int\frac{dx}{x\sqrt{1-\log^2x}}=\text{arc sin}\,(\log x)+C.$$

16. $$\int\frac{du}{\sqrt{a^2-(u+b)^2}}=\text{arc sin}\,\frac{u+b}{a}+C.$$

17. $$\int\frac{a\,dz}{(z-e)^2+b^2}=\frac{a}{b}\,\text{arc tan}\,\frac{z-e}{b}+C.$$

18. $$\int\frac{dx}{x^2+2\,x+5}=\frac{1}{2}\,\text{arc tan}\,\frac{x+1}{2}+C.$$

*Hint.* By completing the square in the denominator this expression may be brought to a form similar to that of Ex. 17. Thus,

$$\int\frac{dx}{x^2+2\,x+5}=\int\frac{dx}{(x^2+2\,x+1)+4}=\int\frac{dx}{(x+1)^2+4}=\frac{1}{2}\,\text{arc tan}\,\frac{x+1}{2}+C. \qquad \text{By [18]}$$

Here $v=x+1$ and $a=2$.

19. $$\int\frac{dx}{\sqrt{2+x-x^2}}=\text{arc sin}\,\frac{2\,x-1}{3}+C.$$

*Hint.* Bring this to the form of Ex. 16 by completing the square. Thus,

$$\int\frac{dx}{\sqrt{2+x-x^2}}=\int\frac{dx}{\sqrt{2-(x^2-x)}}=\int\frac{dx}{\sqrt{2-(x^2-x+\frac{1}{4})+\frac{1}{4}}}=\int\frac{dx}{\sqrt{\frac{9}{4}-(x-\frac{1}{2})^2}}=\text{arc sin}\,\frac{2\,x-1}{3}+C. \qquad \text{By [20]}$$

Here $r=x-\frac{1}{2}$ and $a=\frac{3}{2}$.

20. $$\int\frac{dx}{1+x+x^2}=\frac{2}{\sqrt{3}}\,\text{arc tan}\,\frac{2\,x+1}{\sqrt{3}}+C.$$

**21.** $\displaystyle\int\frac{dx}{\sqrt{3x-x^2-2}}=\text{arc sin}\,(2x-3)+C.$

**22.** $\displaystyle\int\frac{dv}{v^2-6v+5}=\frac{1}{4}\log\frac{v-5}{v-1}+C.$

**23.** $\displaystyle\int\frac{dy}{y^2+3y+1}=\frac{1}{\sqrt{5}}\log\frac{2y+3-\sqrt{5}}{2y+3+\sqrt{5}}+C.$

**24.** $\displaystyle\int\frac{dt}{\sqrt{1+t+t^2}}=\log\left(t+\tfrac{1}{2}+\sqrt{t^2+t+1}\right)+C.$

**25.** $\displaystyle\int\frac{dz}{2z^2-2z+1}=\text{arc tan}\,(2z-1)+C.$

**26.** $\displaystyle\int\frac{ds}{\sqrt{2as+s^2}}=\log\left(s+a+\sqrt{2as+s^2}\right)+C.$

**27.** $\displaystyle\int\frac{dx}{x\sqrt{c^2x^2-a^2b^2}}=\frac{1}{ab}\,\text{arc sec}\,\frac{cx}{ab}+C.$

**28.** $\displaystyle\int\frac{3x^2dx}{\sqrt{x^3-9x^6}}=\frac{1}{3}\,\text{arc vers}\,18x^3+C.$

**29.** $\displaystyle\int\frac{(b+ex)\,dx}{a^2+x^2}=\frac{b}{a}\,\text{arc tan}\,\frac{x}{a}+\frac{e}{2}\log\left(a^2+x^2\right)+C.$

**30.** $\displaystyle\int\frac{2x-5}{3x^2-2}\,dx=\frac{1}{3}\log\left(3x^2-2\right)-\frac{5}{2\sqrt{6}}\log\frac{x\sqrt{3}-\sqrt{2}}{x\sqrt{3}+\sqrt{2}}+C.$

**184. Trigonometric differentials.** We shall now consider some trigonometric differentials of frequent occurrence which may be readily integrated by being transformed into standard forms by means of simple trigonometric reductions.

**Example I.** *To find* $\int\sin^m x\cos^n x\,dx$.

When either $m$ or $n$ is a positive odd integer, no matter what the other may be, this integration may be performed by means of formula [4],

$$\int v^n dv=\frac{v^{n+1}}{n+1}.$$

For the integral is reducible to the form

$$\int(\textit{terms involving only}\cos x)\sin x\,dx,$$

when $\sin x$ has the odd exponent, and to the form

$$\int(\textit{terms involving only}\sin x)\cos x\,dx,$$

when $\cos x$ has the odd exponent. We shall illustrate this by means of examples.

Ex. 1. Find $\int \sin^2 x \cos^5 x dx$.

*Solution.* $\int \sin^2 x \cos^5 x dx = \int \sin^2 x \cos^4 x \cos x dx$

$= \int \sin^2 x (1 - \sin^2 x)^2 \cos x dx$ **28**, p. 2

$= \int (\sin^2 x - 2 \sin^4 x + \sin^6 x) \cos x dx$

$= \int (\sin x)^2 \cos x dx - 2\int (\sin x)^4 \cos x dx + \int (\sin x)^6 \cos x dx$

$= \frac{\sin^3 x}{3} - \frac{2 \sin^5 x}{5} + \frac{\sin^7 x}{7} + C.$ By *[4]*

Here $v = \sin x$, $dv = \cos x dx$, and $n = 2, 4$, and $6$ respectively.

Ex. 2. Find $\int \cos^3 x dx$.

*Solution.* $\int \cos^3 x dx = \int \cos^2 x \cos x dx = \int (1 - \sin^2 x) \cos x dx$

$= \int \cos x dx - \int \sin^2 x \cos x dx$

$= \sin x - \frac{\sin^3 x}{3} + C.$

## EXAMPLES

1. $\int \sin^3 x dx = \frac{1}{3} \cos^3 x - \cos x + C.$ 2. $\int \sin^2 x \cos x dx = \frac{\sin^3 x}{3} + C.$

3. $\int \sin^5 x dx = - \cos x + \frac{2}{3} \cos^3 x - \frac{\cos^5 x}{5} + C.$

4. $\int \cos^5 x dx = \sin x - \frac{2}{3} \sin^3 x + \frac{\sin^5 x}{5} + C.$

5. $\int \sin s \cos s ds = \frac{\sin^2 s}{2} + C.$

6. $\int \cos^4 x \sin^3 x dx = - \frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x + C.$

7. $\int \cos^2 a \sin a da = - \frac{\cos^3 a}{3} + C.$

8. $\int \frac{\cos^3 x dx}{\sin^4 x} = \csc x - \frac{1}{3} \csc^3 x + C.$ 9. $\int \frac{\sin^3 a da}{\cos^2 a} = \sec a + \cos a + C.$

10. $\int \sin^{\frac{3}{7}} \phi \cos^3 \phi d\phi = \frac{7}{10} \sin^{\frac{10}{7}} \phi - \frac{7}{24} \sin^{\frac{24}{7}} \phi + C.$

11. $\int \sin^{\frac{2}{3}} \theta \cos^5 \theta d\theta = \frac{3}{5} \sin^{\frac{5}{3}} \theta - \frac{6}{11} \sin^{\frac{11}{3}} \theta + \frac{3}{17} \sin^{\frac{17}{3}} \theta + C.$

12. $\int \frac{\sin^5 y}{\sqrt{\cos y}} dy = -2\sqrt{\cos y}\,(1 - \frac{2}{5}\cos^2 y + \frac{1}{9}\cos^4 y) + C.$

13. $\int \frac{\cos^5 t\,dt}{\sqrt[3]{\sin t}} = \frac{3}{2}\sin^{\frac{2}{3}} t\,(1 - \frac{1}{2}\sin^2 t + \frac{1}{7}\sin^4 t) + C.$

**Example II.** *To find* $\int \tan^n x dx$, *or* $\int \cot^n x dx$.

These forms can be readily integrated, when $n$ is an integer, on somewhat the same plan as the previous examples.

**Ex. 1.** Find $\int \tan^4 x dx$.

*Solution.*
$$\int \tan^4 x dx = \int \tan^2 x(\sec^2 x - 1)dx \qquad 28, \text{ p. } 2$$
$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$
$$= \int (\tan x)^2 d(\tan x) - \int (\sec^2 x - 1)dx$$
$$= \frac{\tan^3 x}{3} - \tan x + x + C.$$

**Example III.** *To find* $\int \sec^n x dx$, *or* $\int \csc^n x dx$.

These can be easily integrated when $n$ is an even positive integer.

**Ex. 2.** Find $\int \sec^6 x dx$.

*Solution.*
$$\int \sec^6 x dx = \int (\tan^2 x + 1)^2 \sec^2 x dx \qquad 28, \text{ p. } 2$$
$$= \int (\tan x)^4 \sec^2 x dx + 2\int (\tan x)^2 \sec^2 x dx + \int \sec^2 x dx$$
$$= \frac{\tan^5 x}{5} + 2\frac{\tan^3 x}{3} + \tan x + C.$$

**Example IV.** *To find* $\int \tan^m x \sec^n x dx$, *or* $\int \cot^m x \csc^n x dx$.

When $n$ is a positive even number we proceed as in Example III.

**Ex. 3.** Find $\int \tan^6 x \sec^4 x dx$.

*Solution.*
$$\int \tan^6 x \sec^9 x dx = \int \tan^6 x(\tan^2 x + 1)\sec^7 x dx \qquad 28, \text{ p. } 2$$
$$= \int (\tan x)^8 \sec^2 x dx + \int \tan^6 x \sec^7 x dx$$
$$= \frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C. \qquad \text{By } [4]$$

Here $v = \tan x$, $dv = \sec^2 x dx$, etc.

When $m$ is odd we may proceed as in the following example.

**Ex. 4.** Find $\int \tan^5 x \sec^3 x dx = \int \tan^4 x \sec^2 x \sec x \tan x dx$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x dx \qquad \text{28, p.}$$

$$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x) \sec x \tan x dx$$

$$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C. \qquad \text{By [4}$$

Here $v = \sec x$, $dv = \sec x \tan x dx$, etc.

## EXAMPLES

1. $\int \tan^3 x dx = \frac{\tan^2 x}{2} + \log \cos x + C.$

2. $\int \cot^3 x dx = -\frac{\cot^2 x}{2} - \log \sin x + C.$

3. $\int \cot^4 \frac{x}{3} dx = -\cot^3 \frac{x}{3} + 3\cot \frac{x}{3} + x + C.$

4. $\int \cot^2 x dx = -\cot x - x + C.$

5. $\int \cot^5 \alpha d\alpha = -\frac{1}{4}\cot^4 \alpha + \frac{1}{2}\cot^2 \alpha + \log \sin \alpha + C.$

6. $\int \tan^5 \frac{y}{4} dy = \tan^4 \frac{y}{4} - 2\tan^2 \frac{y}{4} + \log \sec \frac{y}{4} + C.$

7. $\int \sec^8 x dx = \frac{\tan^7 x}{7} + \frac{3\tan^5 x}{5} + \tan^3 x + \tan x + C.$

8. $\int \csc^6 x dx = -\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C.$

9. $\int \tan^4 \phi \sec^4 \phi d\phi = \frac{\tan^7 \phi}{7} + \frac{\tan^5 \phi}{5} + C.$

10. $\int \tan^3 \theta \sec^5 \theta d\theta = \frac{1}{7}\sec^7 \theta - \frac{1}{5}\sec^5 \theta + C.$

11. $\int \cot^5 x \csc^4 x dx = -\frac{\cot^6 x}{6} - \frac{\cot^8 x}{8} + C.$

12. $\int \tan^{\frac{3}{2}} x \sec^4 x dx = \frac{2\tan^{\frac{5}{2}} x}{5} + \frac{2\tan^{\frac{9}{2}} x}{9} + C.$

13. $\int \tan^5 y \sec^{\frac{3}{2}} y dy = 2\sec^{\frac{3}{2}} y \left(\frac{\sec^4 y}{11} - \frac{2\sec^2 y}{7} + \frac{1}{3}\right) + C.$

14. $\int \frac{\sec^6 \alpha d\alpha}{\tan^4 \alpha} = \tan \alpha - 2\cot \alpha - \frac{\cot^3 \alpha}{3} + C.$

**15.** $\int (\tan^2 z + \tan^4 z)\, dz = \frac{1}{3}\tan^3 z + C.$

**16.** $\int (\tan t + \cot t)^3\, dt = \frac{1}{2}(\tan^2 t - \cot^2 t) + \log \tan^2 t + C.$

**Example V.** *To find* $\int \sin^m x \cos^n x\, dx$ *by means of multiple angles.*

When either $m$ or $n$ is a positive odd integer the shortest method is that shown in Example I, p. 303. When $m$ and $n$ are both positive even integers the given differential expression may be transformed by suitable trigonometric substitutions into an expression involving sines and cosines of multiple angles, and then integrated. For this purpose we employ the following formulas:

$$\sin u \cos u = \tfrac{1}{2}\sin 2u, \qquad 36, \text{ p. } 2$$

$$\sin^2 u = \tfrac{1}{2} - \tfrac{1}{2}\cos 2u, \qquad 38, \text{ p. } 3$$

$$\cos^2 u = \tfrac{1}{2} + \tfrac{1}{2}\cos 2u. \qquad 39, \text{ p. } 3$$

**Ex. 1.** Find $\int \cos^2 x\, dx$.

*Solution.*

$$\int \cos^2 x\, dx = \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\, dx \qquad 38, \text{ p. } 3$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C.$$

**Ex. 2.** Find $\int \sin^2 x \cos^2 x\, dx$.

*Solution.*

$$\int \sin^2 x \cos^2 x\, dx = \tfrac{1}{4}\int \sin^2 2x\, dx \qquad 36, \text{ p. } 2$$

$$= \tfrac{1}{4}\int (\tfrac{1}{2} - \tfrac{1}{2}\cos 4x)\, dx \qquad 38, \text{ p. } 3$$

$$= \frac{x}{8} - \frac{1}{32}\sin 4x + C.$$

**Ex. 3.** Find $\int \sin^4 x \cos^2 x\, dx$.

*Solution.*

$$\int \sin^4 x \cos^2 x\, dx = \int (\sin x \cos x)^2 \sin^2 x\, dx$$

$$= \int \tfrac{1}{4}\sin^2 2x(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\, dx \qquad 36, \text{ p. } 2;\ 38, \text{ p. } 3$$

$$= \tfrac{1}{8}\int \sin^2 2x\, dx - \tfrac{1}{8}\int \sin^2 2x \cos 2x\, dx$$

$$= \tfrac{1}{8}\int (\tfrac{1}{2} - \tfrac{1}{2}\cos 4x)\, dx - \tfrac{1}{8}\int \sin^2 2x \cos 2x\, dx$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C.$$

**Example VI.** *To find* $\int \sin mx \cos nx dx$, $\int \sin mx \cos nx dx$, or $\int \cos mx \cos nx dx$, *when* $m \neq n$.

By 41, p. 3, $\sin mx \cos nx = \frac{1}{2} \sin (m+n)x + \frac{1}{2} \sin (m-n)x$.

$$\therefore \int \sin mx \cos nx dx = \frac{1}{2}\int \sin (m+n)x dx + \frac{1}{2}\int \sin (m-n) x dx$$

$$= -\frac{\cos (m+n)x}{2(m+n)} - \frac{\cos (m-n)x}{2(m-n)} + C.$$

Similarly we find

$$\int \sin mx \sin nx dx = -\frac{\sin (m+n)x}{2(m+n)} + \frac{\sin (m-n)x}{2(m-n)} + C,$$

$$\int \cos mx \cos nx dx = \frac{\sin (m+n)x}{2(m+n)} + \frac{\sin (m-n)x}{2(m-n)} + C.$$

## EXAMPLES

1. $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C.$
2. $\int \sin^4 x dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$
3. $\int \cos^4 x dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$
4. $\int \sin^6 x dx = \frac{1}{16}\left(5x - 4\sin 2x + \frac{\sin^3 2x}{3} + \frac{3}{4}\sin 4x\right) + C.$
5. $\int \cos^6 x dx = \frac{1}{16}\left(5x + 4\sin 2x - \frac{\sin^3 2x}{3} + \frac{3}{4}\sin 4x\right) + C.$
6. $\int \sin^4 a \cos^2 a da = -\frac{\sin^3 2a}{48} + \frac{a}{16} - \frac{\sin 4a}{64} + C.$
7. $\int \sin^4 t \cos^4 t dt = \frac{1}{128}\left(3t - \sin 4t + \frac{\sin 8t}{8}\right) + C.$
8. $\int \cos^6 x \sin^2 x dx = \frac{1}{128}\left(5x + \frac{8}{3}\sin^3 2x - \sin 4x - \frac{\sin 8x}{8}\right) + C.$
9. $\int \cos 3y \sin 5y dy = -\frac{\cos 8y}{16} - \frac{\cos 2y}{4} + C.$
10. $\int \sin 5z \sin 6z dz = -\frac{\sin 11z}{22} + \frac{\sin z}{2} + C.$
11. $\int \cos 4s \cos 7s ds = \frac{\sin 11s}{22} + \frac{\sin 3s}{6} + C.$
12. $\int \cos \frac{3}{4}x \sin \frac{1}{4}x dx = -\frac{1}{2}\cos x + \cos \frac{1}{2}x + C.$
13. $\int \cos 3x \cos \frac{4}{3}x dx = \frac{3}{26}\sin \frac{13}{3}x + \frac{3}{10}\sin \frac{5}{3}x + C.$

# CHAPTER XXV

## CONSTANT OF INTEGRATION

**185. Determination of the constant of integration by means of initial conditions.** As was pointed out on p. 290, the constant of integration may be found in any given case when we know the value of the integral for some value of the variable. In fact it is necessary, in order to be able to determine the constant of integration, to have some data given in addition to the differential expression to be integrated. Let us illustrate this by means of an example.

**Ex. 1.** Find a function whose first derivative is $3x^2 - 2x + 5$, and which shall have the value 12 when $x = 1$.

*Solution.* $(3x^2 - 2x + 5)\,dx$ is the differential expression to be integrated. Thus,

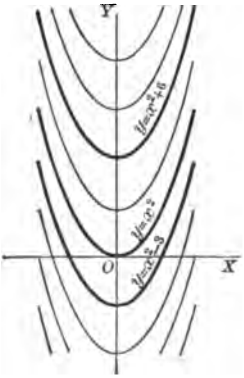
$$\int (3x^2 - 2x + 5)\,dx = x^3 - x^2 + 5x + C,$$

where $C$ is the constant of integration. From the conditions of our problem this result must equal 12 when $x = 1$; that is,

$$12 = 1 - 1 + 5 + C, \text{ or, } C = 7.$$

Hence $x^3 - x^2 - 5x + 7$ is the required function.

**186. Geometrical signification of the constant of integration.** We shall illustrate this by means of examples.



**Ex. 1.** Determine the equation of the curve at every point of which the tangent has the slope $2x$.

*Solution.* Since the slope of the tangent to a curve at any point is $\frac{dy}{dx}$, we have by hypothesis

$$\frac{dy}{dx} = 2x,$$

or, $$dy = 2x\,dx.$$

Integrating, $$y = 2\int x\,dx, \text{ or,}$$

(*A*) $$y = x^2 + C,$$

where $C$ is the constant of integration. Now if we give to $C$ a series of values, say 6, 0, $-3$, (*A*) yields the equations

$$y = x^2 + 6,\ y = x^2,\ y = x^2 - 3,$$

whose loci are parabolas with axes coinciding with the axis of $y$ and having 6, 0, $-3$ respectively as intercepts on the axis of $Y$.

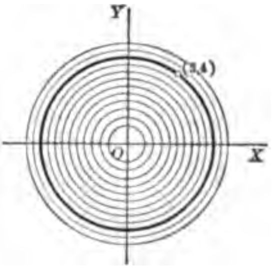
All of the parabolas ($A$) (there are an infinite number of them) have the same value of $\frac{dy}{dx}$; that is, they have the same direction (or slope) for the same value of $x$. It will also be noticed that the difference in the lengths of their ordinates remains the same for all values of $x$. Hence all the parabolas can be obtained by moving any one of them vertically up or down, the value of $C$ in this case not affecting the slope of the curve.

If in the above example we impose the additional condition that the curve shall pass through the point (1, 4), then the coördinates of this point must satisfy ($A$), giving

$$4 = 1 + C, \text{ or, } C = 3.$$

Hence the particular curve required is the parabola $y = x^2 + 3$.

**Ex. 2.** Determine the equation of a curve such that the slope of the tangent to the curve at any point is the negative ratio of the abscissa to the ordinate.



*Solution.* The condition of the problem is expressed by the equation

$$\frac{dy}{dx} = -\frac{x}{y},$$

or, separating the variables,

$$ydy = -xdx.$$

Integrating, $$\frac{y^2}{2} = -\frac{x^2}{2} + C,$$

or, $$x^2 + y^2 = 2C.$$

This we see represents a series of concentric circles with their centers at the origin.

If, in addition, we impose the condition that the curve must pass through the point (3, 4), then

$$9 + 16 = 2C.$$

Hence the particular curve required is the circle $x^2 + y^2 = 25$.

**187. Physical signification of the constant of integration.** The following examples will illustrate what is meant.

**Ex. 1.** Find the laws governing the motion of a point which moves in a straight line with constant acceleration.

*Solution.* Since the acceleration $\left[= \frac{dv}{dt} \text{ from (14), p. 105}\right]$ is constant, say $f$, we have

$$\frac{dv}{dt} = f,$$

or, $$dv = fdt. \text{ Integrating,}$$

($A$) $$v = ft + C.$$

To determine $C$, suppose that the *initial* velocity be $v_0$; that is, let

$$v = v_0 \text{ when } t = 0.$$

These values substituted in (*A*) give

$$v_0 = 0 + C, \text{ or, } C = v_0.$$

Hence (*A*) becomes

(*B*) $$v = ft + v_0.$$

Since $v = \frac{ds}{dt}$ $\left[(9), \text{p. } 103\right]$, we get from (*B*)

$$\frac{ds}{dt} = ft + v_0,$$

or, $$ds = ftdt + v_0dt. \quad \text{Integrating,}$$

(*C*) $$s = \tfrac{1}{2}ft^2 + v_0t + C.$$

To determine *C*, suppose that the *initial* space (= distance) be $s_0$; that is, let

$$s = s_0 \text{ when } t = 0.$$

These values substituted in (*C*) give

$$s_0 = 0 + 0 + C, \text{ or, } C = s_0.$$

Hence (*C*) becomes

(*D*) $$s = \tfrac{1}{2}ft^2 + v_0t + s_0.$$

By substituting the values $f = g$, $v_0 = 0$, $s_0 = 0$, $s = h$ in (*B*) and (*D*) we get the laws of motion of a body falling from rest in a vacuum, namely,

(*Ba*) $$v = gt, \text{ and}$$

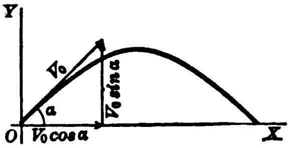
(*Da*) $$h = \tfrac{1}{2}gt^2.$$

Eliminating *t* between (*Ba*) and (*Da*) gives

$$v = \sqrt{2gh}.$$

Ex. 2. Discuss the motion of a projectile having an initial velocity $v_0$ inclined an angle $a$ with the horizontal, the resistance of the air being neglected.

*Solution.* Assume the *XY* plane as the plane of motion, *OX* as horizontal, and *OY* as vertical, and let the projectile be thrown from the origin.



Suppose the projectile to be acted upon by gravity alone. Then the acceleration in the horizontal direction will be zero and in the vertical direction $-g$. Hence from (15), p. 105,

$$\frac{dv_x}{dt} = 0 \text{ and } \frac{dv_y}{dt} = -g.$$

Integrating, $$v_x = C_1 \text{ and } v_y = -gt + C_2.$$

But $v_0 \cos a$ = initial velocity in the horizontal direction,

and $v_0 \sin a$ = initial velocity in the vertical direction.

Hence $$C_1 = v_0 \cos a \text{ and } C_2 = v_0 \sin a, \text{ giving}$$

(*E*) $$v_x = v_0 \cos a \text{ and } v_y = -gt + v_0 \sin a.$$

But from (10) and (11), p. 104, $v_x = \frac{dx}{dt}$ and $v_y = \frac{dy}{dt}$; therefore (*E*) gives

$$\frac{dx}{dt} = v_0 \cos a \text{ and } \frac{dy}{dt} = -gt + v_0 \sin a,$$

or, $$dx = v_0 \cos adt \text{ and } dy = -gtdt + v_0 \sin adt.$$

Integrating, we get

(*F*) $$x = v_0 \cos a \cdot t + C_3 \text{ and } y = -\tfrac{1}{2} gt^2 + v_0 \sin a \cdot t + C_4.$$

To determine $C_3$ and $C_4$, we observe that when

$$t = 0,\ x = 0 \text{ and } y = 0.$$

Substituting these values in (*F*) gives

$$C_3 = 0 \text{ and } C_4 = 0.$$

Hence

(*G*) $$x = v_0 \cos a \cdot t, \text{ and}$$

(*H*) $$y = -\tfrac{1}{2} gt^2 + v_0 \sin a \cdot t.$$

Eliminating $t$ between (*G*) and (*H*), we obtain

(*I*) $$y = x \tan a - \frac{gx^2}{2 v_0^2 \cos^2 a},$$

which is the equation of the *trajectory*, and shows that the projectile will move in a parabola.

## EXAMPLES

Find the function whose first derivative is

1. $x - 3$, knowing that the function equals 9 when $x = 2$. *Ans.* $\frac{x^2}{2} - 3x + 13.$

2. $3 + x - 5x^2$, knowing that the function equals $-20$ when $x = 6$. *Ans.* $124 + 3x + \frac{x^2}{2} - \frac{5x^3}{3}.$

3. $(y^3 - b^2y)\,dy$, knowing that the function equals 0 when $y = 2$. *Ans.* $\frac{y^4}{4} - \frac{b^2y^2}{2} + 2b^2 - 4.$

4. $\sin a + \cos a$, knowing that the function equals 2 when $a = \frac{\pi}{2}$. *Ans.* $\sin a - \cos a + 1.$

5. $\frac{dt}{t} - \frac{dt}{2 - t}$, knowing that the function equals 0 when $t = 1$. *Ans.* $\log(2t - t^2).$

Find the equation of a curve such that the slope of the tangent at any point is

6. $3x - 2$. *Ans.* $y = \frac{3x^2}{2} - 2x + C.$

7. $xy$. *Ans.* $y = ce^{\frac{x^2}{2}}.$

8. $x^2 + 5x$, the curve passing through the point (0, 3). *Ans.* $y = \frac{x^3}{3} + \frac{5x^2}{2} + 3.$

9. $\frac{p}{y}$, the curve passing through the point (0, 0). *Ans.* $y^2 = 2px.$

10. $\frac{b^2x}{a^2y}$, the curve passing through the point $(a, 0)$. *Ans.* $b^2x^2 - a^2y^2 = a^2b^2.$

11. $m$, the curve making an intercept $b$ on the axis of $y$. *Ans.* $y = mx + b.$

Find the relation between $x$ and $y$, knowing that

**12.** $\frac{dy}{dx} = \frac{x^2}{y}$. *Ans.* $\frac{y^2}{2} = \frac{x^3}{3} + C$.

**13.** $xdy + ydx = 0$. *Ans.* $xy = C$.

**14.** $\frac{dy}{dx} = \frac{1+x}{1-y}$, if $y = 0$ when $x = 0$. *Ans.* $x^2 + y^2 + 2x - 2y = 0$.

**15.** $(1-y)dx + (1+x)dy = 0$. *Ans.* $\log\frac{1+x}{1-y} = C$.

**16.** Find the equation of the curve whose subnormal is constant and equal to $2a$.
*Ans.* $y^2 = 4ax + C$, a parabola.

*Hint.* From (4), p. 90, subnormal $= y\frac{dy}{dx}$.

**17.** Find the curve whose subtangent is constant and equal to $a$ [see (3), p. 90].
*Ans.* $a \log y = x + C$.

**18.** Find the curve whose subnormal equals the abscissa of the point of contact.
*Ans.* $y^2 - x^2 = 2C$, an equilateral hyperbola.

**19.** Find the curve whose normal is constant ($= R$), assuming that $y = R$ when $x = 0$. *Ans.* $x^2 + y^2 = R^2$, a circle.

*Hint.* From (6), p. 90, length of normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, or, $dx = \pm(R^2-y^2)^{-\frac{1}{2}}ydy$.

**20.** Find the curve whose subtangent equals three times the abscissa of the point of contact. *Ans.* $x = cy^3$.

**21.** Show that the curve whose polar subtangent [see (7), p. 99] is constant is the reciprocal spiral.

**22.** Show that the curve whose polar subnormal [see (8), p. 99] is constant is the spiral of Archimedes.

**23.** Find the curve in which the polar subnormal is proportional to the length of the radius vector. *Ans.* $\rho = ce^{a\theta}$.

**24.** Find the curve in which the polar subnormal is proportional to the sine of the vectorial angle. *Ans.* $\rho = c - a\cos\theta$.

**25.** Find the curve in which the polar subtangent is proportional to the length of the radius vector. *Ans.* $\rho = ce^{a\theta}$.

**26.** Determine the curve in which the polar subtangent and the polar subnormal are in a constant ratio. *Ans.* $\rho = ce^{a\theta}$.

**27.** Find the equation of the curve in which the angle between the radius vector and the tangent is one half the vectorial angle. *Ans.* $\rho = c(1 - \cos\theta)$.

Assuming that $v = v_0$ when $t = 0$, find the relation between $v$ and $t$, knowing that the acceleration is

**28.** Zero. *Ans.* $v = v_0$.

**29.** Constant $= k$. *Ans.* $v = v_0 + kt$.

**30.** $a + bt$. *Ans.* $v = v_0 + at + \frac{bt^2}{2}$.

Assuming that $s = 0$ when $t = 0$, find the relation between $s$ and $t$, knowing that the velocity is

**31.** Constant ($= v_0$). *Ans.* $s = v_0 t$.

**32.** $m + nt$. *Ans.* $s = mt + \frac{nt^2}{2}$.

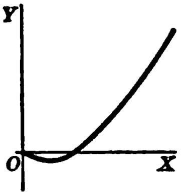
**33.** $3 + 2t - 3t^2$. *Ans.* $s = 3t + t^2 - t^3$.

**34.** The velocity of a body starting from rest is $5t^2$ feet per second after $t$ seconds. (a) How far will it be from the point of starting in 3 seconds? (b) In what time will it pass over a distance of 360 feet measured from the starting point?

*Ans.* (a) 45 ft.; (b) 6 seconds.

**35.** A train starting from a station has after $t$ hours a speed of $t^3 - 21t^2 + 80t$ miles per hour. Find (a) its distance from the station; (b) during what interval the train was moving backwards; (c) when the train repassed the station; (d) the distance the train had traveled when it passed the station the last time.

*Ans.* (a) $\frac{1}{4}t^4 - 7t^3 + 40t^2$ miles; (b) from 5th to 16th hour; (c) in 8 and 20 hours; (d) $4658\frac{1}{2}$ miles.



**36.** A body starts from $O$ and in $t$ seconds its velocity in the $X$ direction is $12t$ and in the $Y$ direction $4t^2 - 9$. Find (a) the distances traversed parallel to each axis; (b) the distance traversed along the path; (c) the equation of the path.

*Ans.* (a) $x = 6t^2$, $y = \frac{4}{3}t^3 - 9t$;

(b) $s = \frac{4}{3}t^3 + 9t$; (c) $y = \left(\frac{2}{9}x - 9\right)\sqrt{\frac{x}{6}}$.

**37.** The equation giving the strength of the current $i$ for the time $t$ after the source of E.M.F. is removed, is ($R$ and $L$ being constants)

$$Ri = -L\frac{di}{dt}.$$

Find $i$, assuming that $I$ = current when $t = 0$. *Ans.* $i = Ie^{-\frac{Rt}{L}}$.

**38.** Find the current of discharge $i$ from a condenser of capacity $C$ in a circuit of resistance $R$, assuming the initial current to be $I_0$, having given the relation

$$\frac{di}{i} = \frac{dt}{CR},$$

$C$ and $R$ being constants. *Ans.* $i = I_0 e^{\frac{t}{CR}}$.

# CHAPTER XXVI

## INTEGRATION OF RATIONAL FRACTIONS

**188. Introduction.** A rational fraction is a fraction the numerator and denominator of which are integral rational functions.* If the degree of the numerator is equal to or greater than that of the denominator, the fraction may be reduced to a mixed quantity by dividing the numerator by the denominator. For example,

$$\frac{x^4+3x^3}{x^2+2x+1}=x^2+x-3+\frac{5x+3}{x^2+2x+1}.$$

The last term is a fraction reduced to its lowest terms, having the degree of the numerator less than that of the denominator. It readily appears that the other terms are at once integrable, and hence we need consider only the fraction.

In order to integrate a differential expression involving such a fraction it is often necessary to resolve it into simpler partial fractions, i.e. to replace it by the algebraic sum of fractions of forms such that we can complete the integration. That this is always possible when the denominator can be broken up into its real prime factors will be shown in the next section.†

**189. Partial fractions.** Consider the rational fraction

$$(A) \qquad \frac{F(x)}{f(x)}$$

reduced to its lowest terms, the degree of the numerator being less than that of the denominator. If $a$ occurs $a$ times as a root of the equation $f(x)=0$, we may write

$$f(x)=(x-a)^a\phi(x),$$

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* That is, the variable is not affected with fractional or negative exponents (see § 25, p. 16).

† Theoretically, the resolution of the denominator into real quadratic and linear factors is always possible when the coefficients are real, that is, such a resolution exists.

where $\phi(x)$ is not divisible by $x-a$. Therefore $\phi(a) \neq 0$ since $a$ is not a root of $\phi(x)=0$. We may then write $(A)$ in the form

$$(B) \qquad \frac{F(x)}{(x-a)^{\alpha}\phi(x)}.$$

The equation

$$\frac{F(x)}{f(x)}=\frac{A}{(x-a)^{\alpha}}+\frac{F(x)}{(x-a)^{\alpha}\phi(x)}-\frac{A}{(x-a)^{\alpha}}$$

is evidently true for any value of $A$.

Combining the last two fractions,

$$(C) \qquad \frac{F(x)}{f(x)}=\frac{A}{(x-a)^{\alpha}}+\frac{F(x)-A\phi(x)}{(x-a)^{\alpha}\phi(x)}.$$

Now let us determine the value of $A$ so that the numerator of the last fraction in $(C)$ shall be divisible by $x-a$. In that case $x=a$ will be a root of $F(x)-A\phi(x)=0$, and hence

$$F(a)-A\phi(a)=0, \text{ or, } A=\frac{F(a)}{\phi(a)}.$$

This value of $A$ is finite since $\phi(a)\neq 0$.

Having now determined the value of $A$ so that $x-a$ shall be a factor of $F(x)-A\phi(x)$, we may write

$$F(x)-A\phi(x)=(x-a)F_1(x),$$

where the degree of the new function $F_1(x)$ is less than the degree of $(x-a)^{\alpha-1}\phi(x)$.

Hence from $(C)$

$$(D) \qquad \frac{F(x)}{f(x)}=\frac{A}{(x-a)^{\alpha}}+\frac{F\ (x)}{(x-a)^{\alpha-1}\phi(x)}.$$

If $\alpha>1$, we proceed in the same manner with the second fraction on the right-hand side of $(D)$, giving, say,

$$(E) \qquad \frac{F_1(x)}{(x-a)^{\alpha-1}\phi(x)}=\frac{A_1}{(x-a)^{\alpha-1}}+\frac{F_2(x)}{(x-a)^{\alpha-2}\phi(x)},$$

where $A_1=\dfrac{F_1(a)}{\phi(a)}$.

Continuing this process, we get

$$(F)\qquad \frac{F(x)}{f(x)}=\frac{A}{(x-a)^{\alpha}}+\frac{A_1}{(x-a)^{\alpha-1}}+\cdots+\frac{A_{\alpha-1}}{x-a}+\frac{G(x)}{\phi(x)},$$

where $G(x)$ is of lower degree than $\phi(x)$, and $\phi(x)$ does not contain the factor $x-a$.

If now $b$ occurs $\beta$ times as a root of $f(x)=0$, it will also occur $\beta$ times as a root of $\phi(x)=0$, and we may break up

$$\frac{G(x)}{\phi(x)}$$

into a sum of partial fractions in exactly the same way as we decomposed the original fraction.

From the preceding discussion it follows that if $a, b, c, \cdots, l$ are roots of the equation $f(x)=0$, occurring respectively $\alpha, \beta, \gamma, \cdots, \lambda$ times, the given rational fraction may be broken up into a sum of rational fractions as follows:

$$(G)\qquad \begin{aligned}\frac{F(x)}{f(x)}&=\frac{A}{(x-a)^{\alpha}}+\frac{A_1}{(x-a)^{\alpha-1}}+\cdots+\frac{A_{\alpha-1}}{x-a}\\ &+\frac{B}{(x-b)^{\beta}}+\frac{B_1}{(x-b)^{\beta-1}}+\cdots+\frac{B_{\beta-1}}{x-b}\\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ &+\frac{L}{(x-l)^{\lambda}}+\frac{L_1}{(x-l)^{\lambda-1}}+\cdots+\frac{L_{\lambda-1}}{x-l}.\end{aligned}$$

Since every equation of degree $n$ has $n$ roots, it is evident that there will be as many partial fractions as there are units in the degree of $f(x)$.

Instead of finding the constants $A, A_1, \cdots, B, B_1, \cdots, L, L_1, \cdots$ by the method indicated above, it is more usual to clear $(G)$ of fractions. This makes $F(x)$ identically equal to a polynomial in $x$ of degree not greater than $n-1$ [assuming $n$ as the degree of $F(x)$] and therefore containing at most $n$ terms. Since this identity holds true for all values of $x$, we equate the constant terms and the coefficients of like powers of $x$ on both sides. This gives $n$ independent and consistent equations, linear in the constants required. Solving these $n$ simultaneous equations we get the $n$ constants $A, A_1, \cdots, B, B_1, \cdots, L, L_1, \cdots$.

**190. Imaginary roots.** Equation ($G$) on p. 317 holds true when some or all of the roots $a, b, c, \cdots, l$ are complex (see § 11, p. 9). When the coefficients of $F(x)$ and $f(x)$ are real — and this is the only case we shall consider — we may avoid the complex numbers and put our results in real form as follows.

Suppose, for instance, that the root $b$ is complex. It may be written in the form

$$b = g + hi,$$

where $g$ and $h$ are real numbers and $i = \sqrt{-1}$. Then there must be a second complex root, say $c$, conjugate to $b$, namely,

$$c = g - hi.$$

If $b$ occurs $\beta$ times and $c$ occurs $\gamma$ times, we see that $\beta$ must equal $\gamma$. From the manner in which $B, B_1, \cdots, C, C_1, \cdots$ were determined it is evident that if

$$B = G + Hi,\ B_1 = G_1 + H_1 i, \cdots,$$

we shall have

$$C = G - Hi,\ C_1 = G_1 - H_1 i, \cdots.$$

Hence the sum

$$(A) \qquad \frac{B}{(x-b)^{\beta}} + \frac{C}{(x-c)^{\beta}} + \frac{B_1}{(x-b)^{\beta-1}} + \frac{C_1}{(x-c)^{\beta-1}} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad + \frac{B_{\beta-1}}{x-b} + \frac{C_{\beta-1}}{x-c}$$

may be expressed in the real form

$$(B) \qquad \frac{\psi(x)}{[(x-g)^2 + h^2]^{\beta}},$$

where the denominator is of degree $2\beta$ and the numerator of degree not greater than $2\beta - 1$. Let $\psi_1(x)$ be the quotient and $P_1 x + Q_1$ the remainder found in dividing $\psi(x)$ by $[(x-g)^2 + h^2]$. Then

$$\psi(x) = [(x-g)^2 + h^2]\psi_1(x) + P_1 x + Q_1,$$

and ($B$) may be written in the form

$$(C) \qquad \frac{\psi(x)}{[(x-g)^2 + h^2]^{\beta}} = \frac{P_1 x + Q_1}{[(x-g)^2 + h^2]^{\beta}} + \frac{\psi_1(x)}{[(x-g)^2 + h^2]^{\beta-1}}.$$

Similarly the second fraction on the right-hand side of ($C$) may be written in the form

$$(D)\qquad \frac{\psi_1(x)}{[(x-g)^2+h^2]^{\beta-1}}=\frac{P_2x+Q_2}{[(x-g)^2+h^2]^{\beta-1}}+\frac{\psi_2(x)}{[(x-g)^2+h^2]^{\beta-2}}.$$

Continuing this process, ($A$) gives finally

$$\begin{aligned}(E)\qquad &\frac{B}{(x-b)^\beta}+\frac{B_1}{(x-b)^{\beta-1}}+\cdots+\frac{B_{\beta-1}}{x-b}\\ &+\frac{C}{(x-c)^\beta}+\frac{C_1}{(x-c)^{\beta-1}}+\cdots+\frac{C_{\beta-1}}{x-c}\\ &=\frac{P_1x+Q_1}{[(x-g)^2+h^2]^\beta}+\frac{P_2x+Q_2}{[(x-g)^2+h^2]^{\beta-1}}+\cdots+\frac{P_\beta x+Q_\beta}{(x-g)^2+h^2}.\end{aligned}$$

The coefficients $P_1, Q_1, P_2, Q_2, \cdots, P_\beta, Q_\beta$ are calculated by the same method we employed to calculate the coefficients in the last section.

We shall now proceed to illustrate what has been said by working out numerous examples in detail. It is convenient to classify our problems under the following four heads.

Case I. When the roots of $f(x)=0$ are all real and none repeated. The denominator may then be broken up into real linear factors, none of which are repeated.

Case II. When the roots of $f(x)=0$ are all real but some repeated. Then the denominator may be broken up into real linear factors, some of which are repeated.

Case III. When the equation $f(x)=0$ has some imaginary roots, none of which are repeated. Then the denominator may be broken up into a product of real prime factors, there being a real quadratic factor (factor of the second degree) corresponding to each pair of conjugate imaginary roots.

Case IV. When the equation $f(x)=0$ has some imaginary roots repeated. The corresponding quadratic factors are then repeated in the denominator.

When we speak of factors of the denominator we shall mean only real prime factors, as these include all the types that can occur.*

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* If the coefficient of the highest power of the variable in the denominator is different from unity, we begin by dividing both numerator and denominator by this coefficient.

**191. Case I.** *When the factors of the denominator are all of the first degree and none repeated.*

It is evident from $(G)$, p. 317, that to each non-repeated linear factor, such as $x - a$, there corresponds a partial fraction of the form

$$\frac{A}{x-a}.$$

Such a partial fraction may be integrated at once as follows:

$$\int\frac{A\,dx}{x-a} = A\int\frac{dx}{x-a} = A\log(x-a) + C.$$

Ex. 1. Find $\int\frac{(2x+3)\,dx}{x^3+x^2-2x}$.

*Solution.* The factors of the denominator being $x$, $x - 1$, $x + 2$, we assume*

$$(A)\qquad \frac{2x+3}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2},$$

where $A$, $B$, $C$ are constants to be determined.

Clearing $(A)$ of fractions, we get

$$(B)\qquad \begin{aligned} 2x+3 &= A(x-1)(x+2) + B(x+2)x + C(x-1)x \\ &= (A+B+C)x^2 + (A+2B-C)x - 2A. \end{aligned}$$

Since this equation is an identity, we equate the coefficients of the like powers of $x$ in the two members according to the method of Undetermined Coefficients, and obtain the three simultaneous equations

$$(C)\qquad \begin{cases} A+B+C=0, \\ A+2B-C=2, \\ \quad -2A=3. \end{cases}$$

Solving equations $(C)$, we get

$$A = -\tfrac{3}{2},\ B = \tfrac{5}{3},\ C = -\tfrac{1}{6}.$$

Substituting these values in $(A)$,

$$\frac{2x+3}{x(x-1)(x+2)} = -\frac{3}{2x} + \frac{5}{3(x-1)} - \frac{1}{6(x+2)}.$$

$$\begin{aligned} \therefore \int\frac{2x+3}{x(x-1)(x+2)}dx &= -\frac{3}{2}\int\frac{dx}{x} + \frac{5}{3}\int\frac{dx}{x-1} - \frac{1}{6}\int\frac{dx}{x+2} \\ &= -\tfrac{3}{2}\log x + \tfrac{5}{3}\log(x-1) - \tfrac{1}{6}\log(x+2) + \log c \\ &= \log\frac{c(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}}(x+2)^{\frac{1}{6}}}. \quad \textit{Ans.} \end{aligned}$$

A shorter method of finding the values of $A$, $B$, and $C$ from $(B)$ is the following:

Let factor $x = 0$; then $3 = -2A$, or, $A = -\frac{3}{2}$.
Let factor $x - 1 = 0$, or $x = 1$; then $5 = 3B$, or, $B = \frac{5}{3}$.
Let factor $x + 2 = 0$, or $x = -2$; then $-1 = 6C$, or, $C = -\frac{1}{6}$.

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* In the process of decomposing the fractional part of the given differential neither the integral sign nor $dx$ enters.

### EXAMPLES

1. $\displaystyle\int\frac{(x-1)\,dx}{x^2+6x+8}=\log\frac{c(x+4)^{\frac{5}{2}}}{(x+2)^{\frac{3}{2}}}.$

2. $\displaystyle\int\frac{(3x-1)\,dx}{x^2+x-6}=\log[c(x+3)^2(x-2)].$

3. $\displaystyle\int\frac{(x^2+x-1)\,dx}{x^3+x^2-6x}=\log\sqrt[6]{x(x-2)^3(x+3)^2}+C.$

4. $\displaystyle\int\frac{x^5+x^4-8}{x^3-4x}\,dx=\frac{x^3}{3}+\frac{x^2}{2}+4x+\log\frac{x^2(x-2)^5}{(x+2)^3}+C.$

5. $\displaystyle\int\frac{x^4dx}{(x^2-1)(x+2)}=\frac{x^2}{2}-2x+\frac{1}{6}\log\frac{x-1}{(x+1)^3}+\frac{16}{3}\log(x+2)+C.$

6. $\displaystyle\int\frac{(a-b)\,y\,dy}{y^2-(a+b)y+ab}=\log\frac{(y-a)^a}{(y-b)^b}+C.$

7. $\displaystyle\int\frac{(t^2+pq)\,dt}{t(t-p)(t+q)}=\log\frac{(t-p)(t+q)}{t}+C.$

8. $\displaystyle\int\frac{(2z^2-5)\,dz}{z^4-5z^2+6}=\frac{1}{2\sqrt{2}}\log\frac{z-\sqrt{2}}{z+\sqrt{2}}+\frac{1}{2\sqrt{3}}\log\frac{z-\sqrt{3}}{z+\sqrt{3}}+C.$

**192. Case II.** *When the factors of the denominator are all of the first degree and some repeated.*

From ($G$), p. 317, it is clear that to every $n$-fold linear factor, such as $(x-a)^n$, there correspond the $n$ partial fractions

$$\frac{A}{(x-a)^n}+\frac{B}{(x-a)^{n-1}}+\cdots+\frac{L}{x-a}.$$

The last one is integrated as in Case I. The rest are all integrated by means of the power formula. Thus,

$$\int\frac{A\,dx}{(x-a)^n}=A\int(x-a)^{-n}dx=\frac{A}{(1-n)(x-a)^{n-1}}+C.$$

Ex. 1. Find $\displaystyle\int\frac{x^3+1}{x(x-1)^3}dx.$

*Solution.* Since $x-1$ occurs three times as a factor, we assume

$$\frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{(x-1)^3}+\frac{C}{(x-1)^2}+\frac{D}{x-1}.$$

Clearing of fractions,

$$\begin{aligned}x^3+1&=A(x-1)^3+Bx+Cx(x-1)+Dx(x-1)^2\\&=(A+D)x^3+(-3A+C-2D)x^2+(3A+B-C+D)x-A.\end{aligned}$$

Equating the coefficients of like powers of $x$, we get the simultaneous equations

$$\begin{aligned} A + D &= 1, \\ -3A + C - 2D &= 0, \\ 3A + B - C + D &= 0, \\ -A &= 1. \end{aligned}$$

Solving, $A = -1$, $B = 2$, $C = 1$, $D = 2$, and

$$\frac{x^3+1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{2}{x-1}.$$

$$\therefore \int \frac{x^3+1}{x(x-1)^3}dx = -\log x - \frac{1}{(x-1)^2} - \frac{1}{x-1} + 2\log(x-1) + C$$

$$= -\frac{x}{(x-1)^2} + \log\frac{(x-1)^2}{x} + C.$$

## EXAMPLES

1. $\displaystyle\int \frac{x^2dx}{(x+2)^2(x+1)} = \frac{4}{x+2} + \log(x+1) + C.$

2. $\displaystyle\int \frac{(x-8)\,dx}{x^3 - 4x^2 + 4x} = \frac{3}{x-2} + \log\frac{(x-2)^2}{x^2} + C.$

3. $\displaystyle\int \frac{x^2+1}{(x-1)^3}dx = -\frac{1}{(x-1)^2} - \frac{2}{x-1} + \log(x-1) + C.$

4. $\displaystyle\int \frac{(3x+2)\,dx}{x(x+1)^3} = \frac{4x+3}{2(x+1)^2} + \log\frac{x^2}{(x+1)^2} + C.$

5. $\displaystyle\int \frac{x^2dx}{(x+2)^2(x+4)^2} = -\frac{5x+12}{x^2+6x+8} + \log\left(\frac{x+4}{x+2}\right)^2 + C.$

6. $\displaystyle\int \frac{y^2dy}{y^3 + 5y^2 + 8y + 4} = \frac{4}{y+2} + \log(y+1) + C.$

7. $\displaystyle\int \frac{dt}{(t^2-2)^2} = -\frac{t}{4(t^2-2)} + \frac{1}{8\sqrt{2}}\log\frac{t+\sqrt{2}}{t-\sqrt{2}} + C.$

8. $\displaystyle\int \frac{as^2dx}{(s+a)^3} = a\log(s+a) + \frac{2a^2}{s+a} - \frac{a^3}{2(s+a)^2} + C.$

9. $\displaystyle\int \left(\frac{m}{z+m} - \frac{nz}{(z+n)^2}\right)dz = \log(z+m)^m(z+n)^{-n} - \frac{n^2}{z+n} + C.$

**193. Case III.** *When the denominator contains factors of the second degree but none repeated.*

From $(E)$, p. 319, we know that to every non-repeated quadratic factor, such as $x^2 + px + q$, there corresponds a partial fraction of the form

$$\frac{Ax + B}{x^2 + px + q}.$$

This may be integrated as follows:

$$\int \frac{(Ax+B)\,dx}{x^2+px+q} = \int \frac{\left(Ax+\frac{Ap}{2}-\frac{Ap}{2}+B\right)dx}{x^2+px+q}$$

$\left[\text{Adding and subtracting } \frac{Ap}{2} \text{ in the numerator.}\right]$

$$= \int \frac{\left(Ax+\frac{Ap}{2}\right)dx}{x^2+px+q} + \int \frac{\left(-\frac{Ap}{2}+B\right)dx}{x^2+px+q}$$

$$= \frac{A}{2}\int \frac{(2\,x+p)\,dx}{x^2+px+q} + \left(\frac{2\,B-Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)}$$

[Completing the square in the denominator of the second integral.]

$$= \frac{A}{2}\log(x^2+px+q) + \frac{2\,B-Ap}{\sqrt{4\,q-p^2}}\operatorname{arc\,tan}\frac{2\,x+p}{\sqrt{4\,q-p^2}} + C.$$

Since $x^2+px+q=0$ has imaginary roots, we know from 3, p. 1, that $4\,q-p^2>0$.

Ex. 1. Find $\int \frac{4\,dx}{x^3+4\,x}$.

*Solution.* Assume $\qquad \frac{4}{x\,(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$

Clearing of fractions, $4 = A\,(x^2+4) + x\,(Bx+C) = (A+B)\,x^2 + Cx + 4\,A.$
Equating the coefficients of like powers of $x$, we get

$$A+B=0,\quad C=0,\quad 4\,A=4.$$

This gives $A=1,\ B=-1,\ C=0,$ so that $\frac{4}{x\,(x^2+4)} = \frac{1}{x} - \frac{x}{x^2+4}.$

$$\therefore \int \frac{4\,dx}{x\,(x^2+4)} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+4}$$

$$= \log x - \frac{1}{2}\log(x^2+4) + \log c = \log\frac{cx}{\sqrt{x^2+4}}. \quad \textit{Ans.}$$

## EXAMPLES

1. $\int \frac{x\,dx}{(x+1)\,(x^2+4)} = \frac{1}{10}\log\frac{x^2+4}{(x+1)^2} + \frac{2}{5}\operatorname{arc\,tan}\frac{x}{2} + C.$

2. $\int \frac{(2\,x^2-3\,x-3)\,dx}{(x-1)\,(x^2-2\,x+5)} = \log\frac{(x^2-2\,x+5)^{\frac{3}{2}}}{x-1} + \frac{1}{2}\operatorname{arc\,tan}\frac{x-1}{2} + C.$

3. $\int \frac{x^2dx}{1-x^4} = \frac{1}{4}\log\frac{1+x}{1-x} - \frac{1}{2}\operatorname{arc\,tan} x + C.$

4. $\int \frac{dx}{(x^2+1)(x^2+x)} = \frac{1}{4}\log\frac{x^4}{(x+1)^2(x^2+1)} - \frac{1}{2}\arctan x + C.$

5. $\int \frac{(x^3-6)\,dx}{x^4+6x^2+8} = \log\frac{x^2+4}{\sqrt{x^2+2}} + \frac{3}{2}\arctan\frac{x}{2} - \frac{3}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C.$

6. $\int \frac{(5x^2-1)\,dx}{(x^2+3)(x^2-2x+5)} = \log\frac{x^2-2x+5}{x^2+3} + \frac{5}{2}\arctan\frac{x-1}{2} - \frac{2}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C$

7. $\int \frac{dx}{x^3+1} = \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$

8. $\int \frac{(3x-7)\,dx}{x^3+x^2+4x+4} = \log\frac{x^2+4}{(x+1)^2} + \frac{1}{2}\arctan\frac{x}{2} + C.$

9. $\int \frac{z^2dx}{z^4+z^2-2} = \frac{1}{6}\log\left(\frac{z-1}{z+1}\right) + \frac{\sqrt{2}}{3}\arctan\frac{z}{\sqrt{2}} + C.$

10. $\int \frac{4\,dt}{t^4+1} = \frac{1}{\sqrt{2}}\log\frac{t^2+t\sqrt{2}+1}{t^2-t\sqrt{2}+1} + \sqrt{2}\arctan\frac{t\sqrt{2}}{1-t^2} + C.$

11. $\int \frac{dy}{1-y^3} = \frac{1}{6}\log\frac{y^2+y+1}{y^2-2y+1} + \frac{1}{\sqrt{3}}\arctan\frac{2y+1}{\sqrt{3}} + C.$

**194. Case IV.** *When the denominator contains factors of the second degree some of which are repeated.*

To every $n$-fold quadratic factor, such as $(x^2+px+q)^n$, there correspond the $n$ partial fractions

$$(A)\qquad \frac{Ax+B}{(x^2+px+q)^n} + \frac{Cx+D}{(x^2+px+q)^{n-1}} + \cdots + \frac{Lx+M}{x^2+px+q}.$$

To derive a formula for integrating the first one we proceed as follows:

$$\int \frac{Ax+B}{(x^2+px+q)^n}\,dx = \int \frac{\left(Ax+\frac{Ap}{2}-\frac{Ap}{2}+B\right)dx}{(x^2+px+q)^n}$$

[Adding and subtracting $\frac{Ap}{2}$ in the numerator.]

$$= \int \frac{\left(Ax+\frac{Ap}{2}\right)dx}{(x^2+px+q)^n} + \int \frac{\left(-\frac{Ap}{2}+B\right)dx}{(x^2+px+q)^n}$$

$$= \frac{A}{2}\int (x^2+px+q)^{-n}(2x+p)\,dx + \left(\frac{2B-Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^n}.$$

The first one of these may be integrated by [4]; hence

$$(B)\qquad \int\frac{Ax+B}{(x^2+px+q)^n}dx=\frac{A}{2(1-n)(x^2+px+q)^{n-1}}$$
$$+\left(\frac{2B-Ap}{2}\right)\int\frac{dx}{(x^2+px+q)^n}.$$

Let us now differentiate the function $\dfrac{\left(x+\dfrac{p}{2}\right)}{(x^2+px+q)^{n-1}}$.

Thus,

$$\frac{d}{dx}\left(\frac{x+\dfrac{p}{2}}{(x^2+px+q)^{n-1}}\right)=\frac{1}{(x^2+px+q)^{n-1}}-\frac{2(n-1)\left(x+\dfrac{p}{2}\right)^2}{(x^2+px+q)^n}, \text{ or,}$$

$$(C)\qquad d\left(\frac{x+\dfrac{p}{2}}{(x^2+px+q)^{n-1}}\right)$$
$$=\left(\frac{-(2n-3)}{(x^2+px+q)^{n-1}}+\frac{2(n-1)\left(q-\dfrac{p^2}{4}\right)}{(x^2+px+q)^n}\right)dx.$$

$\left[\text{Since } x^2+px+q=\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right), \text{ and } \left(x+\frac{p}{2}\right)^2=(x^2+px+q)-\left(q-\frac{p^2}{4}\right).\right]$

Integrating both sides of $(C)$,

$$\frac{x+\dfrac{p}{2}}{(x^2+px+q)^{n-1}}=-(2n-3)\int\frac{dx}{(x^2+px+q)^{n-1}}$$
$$+2(n-1)\left(q-\frac{p^2}{4}\right)\int\frac{dx}{(x^2+px+q)^n},$$

or, solving for the last integral,

$$(D)\qquad \int\frac{dx}{(x^2+px+q)^n}=\frac{x+\dfrac{p}{2}}{2(n-1)\left(q-\dfrac{p^2}{4}\right)(x^2+px+q)^{n-1}}$$
$$+\frac{2n-3}{2(n-1)\left(q-\dfrac{p^4}{4}\right)}\int\frac{dx}{(x^2+px+q)^{n-1}}.$$

Substituting this result in the second member of ($B$), we get[*]

$$(E)\quad \int\frac{Ax+B}{(x^2+px+q)^n}=\frac{A(p^2-4q)+(2B-Ap)(2x+p)}{2(n-1)(4q-p^2)(x^2+px+q)^{n-1}}$$
$$+\frac{(2B-Ap)(2n-3)}{(n-1)(4q-p^4)}\int\frac{dx}{(x^2+px+q)^{n-1}}.$$

It is seen that our integral has been made to depend on the integration of a rational fraction of the same type in which, however, the quadratic factor occurs only $n-1$ times. By applying the formula ($E$) $n-1$ times successively it is evident that our integral may be made ultimately to depend on

$$\int\frac{dx}{x^2+px+q},$$

and this may be integrated by completing the square, as shown on p. 302.

In the same manner all but the last fraction of ($A$) may be integrated. But this last fraction, namely,

$$\frac{Lx+M}{x^2+px+q},$$

may be integrated by the method already given under the previous case, p. 323.

Ex. 1. Find $\int\frac{(x^3+x^2+2)\,dx}{(x^2+2)^2}$.

*Solution.* Since $x^2+2$ occurs twice as a factor, we assume

$$\frac{x^3+x^2+2}{(x^2+2)^2}=\frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}.$$

Clearing of fractions, we get

$$\begin{aligned}x^3+x^2+2&=Ax+B+(Cx+D)(x^2+2)\\&=Cx^3+Dx^2+(A+2C)x+B+2D.\end{aligned}$$

Equating the coefficients of like powers of $x$,

$$C=1,\ D=1,\ A+2C=0,\ B+2D=2.$$

This gives $\quad A=-2,\ B=0,\ C=1,\ D=1.$

Hence $\quad \frac{x^3+x^2+2}{(x^2+2)^2}=-\frac{2x}{(x^2+2)^2}+\frac{x+1}{x^2+2}$, and

$$\begin{aligned}\int\frac{(x^3+x^2+2)\,dx}{(x^2+2)^2}&=-\int\frac{2x\,dx}{(x^2+2)^2}+\int\frac{x\,dx}{x^2+2}+\int\frac{dx}{x^2+2}\\&=\frac{1}{x^2+2}+\frac{1}{\sqrt{2}}\operatorname{arc\,tan}\frac{x}{\sqrt{2}}+\frac{1}{2}\log(x^2+2)+C.\end{aligned}$$

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[*] $4q-p^2>0$, since $x^2+px+q=0$ has imaginary roots.

**Ex. 2.** Find $\int \frac{2x^3 + x + 3}{(x^2 + 1)^2} dx.$

*Solution.* Since $x^2 + 1$ occurs twice as a factor, we assume

$$\frac{2x^3 + x + 3}{(x^2 + 1)^2} = \frac{Ax + B}{(x^2 + 1)^2} + \frac{Cx + D}{x^2 + 1}.$$

Clearing of fractions,

$$2x^3 + x + 3 = Ax + B + (A + C)x + B + D.$$

Equating the coefficients of like powers of $x$ and solving, we get

$$A = -1,\ B = 3,\ C = 2,\ D = 0.$$

Hence

$$\int \frac{2x^3 + x + 3}{(x^2 + 1)^2} = \int \frac{-x + 3}{(x^2 + 1)^2} dx + \int \frac{2x dx}{x^2 + 1}$$

$$= \log(x^2 + 1) + \int \frac{-x + 3}{(x^2 + 1)^2} dx.$$

Now apply formula $(E)$, p. 326, to the remaining integral. Here

$$A = -1,\ B = 3,\ p = 0,\ q = 1,\ n = 2.$$

Substituting, we get

$$\int \frac{-x + 3}{(x^2 + 1)^2} dx = \frac{1 + 3x}{2(x^2 + 1)} + \frac{3}{2} \int \frac{dx}{x^2 + 1} = \frac{1 + 3x}{2(x^2 + 1)} + \frac{3}{2} \text{arc tan } x.$$

Therefore

$$\int \frac{2x^3 + x + 3}{(x^2 + 1)^2} = \log(x^2 + 1) + \frac{1 + 3x}{2(x^2 + 1)} + \frac{3}{2} \text{arc tan } x + C.$$

## EXAMPLES

1. $\int \frac{x^3 + x - 1}{(x^2 + 2)^2} dx = \frac{2 - x}{4(x^2 + 2)} + \log(x^2 + 2)^{\frac{1}{2}} - \frac{1}{4\sqrt{2}} \text{arc tan } \frac{x}{\sqrt{2}} + C.$

2. $\int \frac{2x dx}{(1 + x)(1 + x^2)^2} = -\log(x + 1) + \frac{\text{arc tan } x}{2} + \frac{x - 1}{2(x^2 + 1)} + C.$

3. $\int \frac{x^7 + x^5 + x^3 + x}{(x^2 + 2)^2(x^2 + 3)^2} dx = \frac{5}{2(x^2 + 2)} + \frac{10}{x^2 + 3} + \frac{19}{2} \log(x^2 + 2) - 9 \log(x^2 + 3) + C.$

4. $\int \frac{(4x^2 - 8x) dx}{(x - 1)^2(x^2 + 1)^2} = \frac{3x^2 - x}{(x - 1)(x^2 + 1)} + \log \frac{(x - 1)^2}{x^2 + 1} + \text{arc tan } x + C.$

5. $\int \frac{(3x + 2) dx}{(x^2 - 3x + 3)^2} = \frac{13x - 24}{3(x^2 - 3x + 3)} + \frac{26}{3\sqrt{3}} \text{arc tan } \frac{2x - 3}{\sqrt{3}} + C.$

Since a rational function may always be reduced to the quotient of two integral rational functions (§ 26, p. 16), i.e. to a rational fraction, it follows from the preceding sections in this chapter that any rational function whose denominator can be broken up into real quadratic and linear factors may be expressed as the algebraic sum of integral rational functions and partial fractions. The terms of this sum have forms all of which we have shown how to integrate. Hence the

**Theorem.** *The integral of every rational function whose denominator can be broken up into real quadratic and linear factors may be found, and is expressible in terms of algebraic, logarithmic, and inverse-trigonometric functions; that is, in terms of the elementary functions.*

# CHAPTER XXVII

## INTEGRATION BY SUBSTITUTION OF A NEW VARIABLE. RATIONALIZATION

**195. Introduction.** In the last chapter it was shown that all rational functions whose denominators can be broken up into real quadratic and linear factors may be integrated. Of algebraic functions which are *not rational*, that is, such as contain radicals, only a small number, relatively speaking, can be integrated in terms of elementary functions. By substituting a new variable, however, these functions can in some cases be transformed into equivalent functions that are either in the list of standard forms (pp. 292, 293) or else are rational. The method of integrating a function that is not rational by substituting for the old variable such a function of a new variable that the result is a rational function is sometimes called *integration by rationalization.* This is a very important artifice in integration and we will now take up some of the more important cases coming under this head.

**196. Differentials containing fractional powers of $x$ only.**

*Such an expression can be transformed into a rational form by means of the substitution*

$$x = z^n,$$

*where $n$ is the least common denominator of the fractional exponents of $x$.*

For $x$, $dx$, and each radical can then be expressed rationally in terms of $z$.

Ex. 1. Find $\int \frac{x^{\frac{2}{3}} - x^{\frac{1}{4}}}{x^{\frac{1}{2}}} dx.$

*Solution.* Since 12 is the L.C.M. of the denominators of the fractional exponents, we assume

$$x = z^{12}.$$

Here $$dx = 12\,z^{11}dz,\ x^{\frac{2}{3}} = z^8,\ x^{\frac{1}{4}} = z^3,\ x^{\frac{1}{2}} = z^6.$$

$$\therefore \int \frac{x^{\frac{2}{3}} - x^{\frac{1}{4}}}{x^{\frac{1}{2}}} dx = \int \frac{z^8 - z^3}{z^6} 12\,z^{11}dz = 12 \int (z^{13} - z^8)\,dz$$

$$= \tfrac{6}{7}\,z^{14} - \tfrac{4}{3}\,z^9 + C = \tfrac{6}{7}\,x^{\frac{7}{6}} - \tfrac{4}{3}\,x^{\frac{3}{4}} + C.$$

[Substituting back the value of $z$ in terms of $x$, namely, $z = x^{\frac{1}{12}}$.]

The general form of the irrational expression here treated is then

$$R(x^{\frac{1}{n}})\,dx,$$

where $R$ denotes a rational function of $x^{\frac{1}{n}}$.

**197. Differentials containing fractional powers of $a + bx$ only.** *Such an expression can be transformed into a rational form by means of the substitution*

$$a + bx = z^n,$$

*where $n$ is the least common denominator of the fractional exponents of the expression $a + bx$.*

For $x$, $dx$, and each radical can then be expressed rationally in terms of $z$.

Ex. 1. Find $\int \frac{dx}{(1+x)^{\frac{3}{2}} + (1+x)^{\frac{1}{2}}}$.

*Solution.* Assume $1 + x = z^2$;

then $dx = 2\,z\,dz,\ (1+x)^{\frac{3}{2}} = z^3$, and $(1+x)^{\frac{1}{2}} = z$.

$$\therefore \int \frac{dx}{(1+x)^{\frac{3}{2}} + (1+x)^{\frac{1}{2}}} = \int \frac{2\,z\,dz}{z^3 + z} = 2\int \frac{dz}{z^2+1}$$

$$= 2 \operatorname{arc\,tan} z + C = 2 \operatorname{arc\,tan} (1+x)^{\frac{1}{2}} + C,$$

when we substitute back the value of $z$ in terms of $x$.

The general integral treated here has then the form

$$R[x, (a+bx)^{\frac{1}{n}}]\,dx,$$

where $R$ denotes a rational function.

## EXAMPLES

1. $\int \frac{x^{\frac{1}{2}}dx}{x^{\frac{3}{4}} + 1} = \frac{4}{3}x^{\frac{3}{4}} - \frac{4}{3}\log(x^{\frac{3}{4}} + 1) + C.$

2. $\int \frac{x^{\frac{3}{2}} - x^{\frac{1}{3}}}{6\,x^{\frac{1}{4}}}dx = \frac{1}{3}\left(\frac{2}{9}x^{\frac{9}{4}} - \frac{6}{13}x^{\frac{13}{12}}\right) + C.$

3. $\int \frac{x^{\frac{1}{6}} + 1}{x^{\frac{7}{6}} + x^{\frac{5}{4}}}dx = -\frac{6}{x^{\frac{1}{6}}} + \frac{12}{x^{\frac{1}{12}}} + 2\log x - 24\log(x^{\frac{1}{12}} + 1) + C.$

4. $\int \frac{dx}{x^{\frac{5}{8}} - x^{\frac{1}{8}}} = \frac{8}{3}x^{\frac{3}{8}} + 2\log\frac{x^{\frac{1}{8}} - 1}{x^{\frac{1}{8}} + 1} + 4 \operatorname{arc\,tan} x^{\frac{1}{8}} + C.$

5. $\int \frac{3\sqrt{x}\,dx}{2\sqrt{x} - \sqrt[3]{x^2}} = -18\left[\frac{x^{\frac{5}{6}}}{5} + \frac{x^{\frac{2}{3}}}{2} + \frac{4\,x^{\frac{1}{2}}}{3} + 4\,x^{\frac{1}{3}} + 16\,x^{\frac{1}{6}} + 32\log(x^{\frac{1}{6}} - 2)\right] + C.$

6. $\displaystyle\int \frac{y^{\frac{1}{2}} + y^{\frac{1}{7}}}{y^{\frac{8}{7}} + y^{\frac{15}{14}}}\,dy = 14\left[y^{\frac{1}{14}} - \frac{y^{\frac{1}{7}}}{2} + \frac{y^{\frac{3}{14}}}{3} - \frac{y^{\frac{2}{7}}}{4} + \frac{y^{\frac{5}{14}}}{5}\right] + C.$

7. $\displaystyle\int \frac{dx}{x(x+1)^{\frac{1}{2}}} = \log\frac{(x+1)^{\frac{1}{2}} - 1}{(x+1)^{\frac{1}{2}} + 1} + C.$

8. $\displaystyle\int \frac{xdx}{(a+bx)^{\frac{3}{2}}} = \frac{2(2a+bx)}{b^2\sqrt{a+bx}} + C.$ 9. $\displaystyle\int \frac{x^2dx}{(4x+1)^{\frac{5}{2}}} = \frac{6x^2+6x+1}{12(4x+1)^{\frac{3}{2}}} + C.$

10. $\displaystyle\int y\sqrt[3]{a+y}\,dy = \tfrac{3}{28}(4y-3a)(a+y)^{\frac{4}{3}} + C.$

11. $\displaystyle\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx = x + 1 + 4\sqrt{x+1} + 4\log(\sqrt{x+1} - 1) + C.$

12. $\displaystyle\int \frac{dx}{1+\sqrt[3]{x+1}} = \frac{3}{2}(x+1)^{\frac{2}{3}} - 3(x+1)^{\frac{1}{3}} + 3\log(1+\sqrt[3]{x+1}) + C.$

13. $\displaystyle\int \frac{dx}{(x+1)^{\frac{1}{2}} - (x+1)^{\frac{1}{4}}} = 3\{(x+1)^{\frac{1}{2}} + 2(x+1)^{\frac{1}{4}} + 2\log[(x+1)^{\frac{1}{4}} - 1]\} + C.$

### 198. Differentials containing no radical except $\sqrt{a+bx+x^2}$.*

*Such an expression can be transformed into a rational form by means of the substitution*

$$\sqrt{a+bx+x^2} = z - x.$$

For, squaring and solving for $x$,

$$x = \frac{z^2 - a}{b + 2z};\ \text{then}\ dx = \frac{2(z^2 + bz + a)\,dz}{(b+2z)^2};$$

and

$$\sqrt{a+bx+x^2}\,(= z - x) = \frac{z^2 + bz + a}{b + 2z}.$$

Hence $x$, $dx$, and $\sqrt{a+bx+x^2}$ are rational when expressed in terms of $z$.

Ex. 1. Find $\displaystyle\int \frac{dx}{\sqrt{1+x+x^2}}$.

*Solution.* Assume $\sqrt{1+x+x^2} = z - x.$

Squaring and solving for $x$,

$$x = \frac{z^2-1}{2z+1};\ \text{then}\ dx = \frac{2(z^2+z+1)\,dz}{(2z+1)^2},$$

and

$$\sqrt{1+x+x^2}\,(= z - x) = \frac{z^2+z+1}{2z+1}.$$

---

* If the radical is of the form $\sqrt{n+px+qx^2}$, $q>0$, it may be written $\sqrt{q}\sqrt{\frac{n}{q}+\frac{p}{q}x+x^2}$, and therefore comes under the above head, where $a=\frac{n}{q}$, $b=\frac{p}{q}$.

$$\therefore \int \frac{dx}{\sqrt{1+x+x^2}} = \int \frac{\dfrac{2(z^2+z+1)\,dz}{(2z+1)^2}}{\dfrac{z^2+z+1}{2z+1}} = \int \frac{2\,dz}{2z+1} = \log[(2z+1)c]$$

$$= \log[(2x+1+2\sqrt{1+x+x^2})c],$$

when we substitute back the value of $z$ in terms of $x$.

**199. Differentials containing no radical except $\sqrt{a+bx-x^2}$.*** *Such an expression can be transformed into a rational form by means of the substitution*

$$\sqrt{a+bx-x^2}\,[=\sqrt{(x-\alpha)(\beta-x)}] = (x-\alpha)z,$$

*where $x-\alpha$ and $\beta-x$ are real† factors of $a+bx-x^2$.*

For, if $\sqrt{a+bx-x^2} = \sqrt{(x-\alpha)(\beta-x)} = (x-\alpha)z$, by squaring, canceling out $(x-\alpha)$, and solving for $x$, we get

$$x = \frac{\alpha z^2+\beta}{z^2+1};\ \text{then}\ dx = \frac{2(\alpha-\beta)z\,dz}{(z^2+1)},$$

and

$$\sqrt{a+bx-x^2}\,[=(x-\alpha)]z = \frac{(\beta-\alpha)z}{z^2+1}.$$

Hence $x$, $dx$, and $\sqrt{a+bx-x^2}$ are rational when expressed in terms of $z$.

Ex. 1. Find $\displaystyle\int \frac{dx}{\sqrt{2+x-x^2}}$.

*Solution.* Since $2+x-x^2 = (x+1)(2-x)$, we assume

$$\sqrt{(x+1)(2-x)} = (x+1)z.$$

Squaring and solving for $x$, $x = \dfrac{2-z^2}{z^2+1}$.

Hence $dx = \dfrac{-6z\,dz}{(z^2+1)^2}$, and $\sqrt{2+x-x^2}\,[=(x+1)z] = \dfrac{3z}{z^2+1}$.

$$\therefore \int \frac{dx}{\sqrt{2+x-x^2}} = -2\int \frac{dz}{z^2+1} = -2\arctan z + C$$

$$= -2\arctan\sqrt{\frac{2-x}{x+1}} + C,$$

when we substitute back the value of $z$ in terms of $x$.

---

[*] If the radical is of the form $\sqrt{n+px-qx^2}$, $q>0$, it may be written $\sqrt{q}\sqrt{\frac{n}{q}+\frac{p}{q}x-x^2}$, and therefore comes under the above head, where $a=\frac{n}{q}$, $b=\frac{p}{q}$.

[†] If the factors of $a+bx-x^2$ are imaginary, $\sqrt{a+bx-x^2}$ is imaginary for all values of $x$. For, if one of the factors is $x-m+in$, the other must be $-(x-m-in)$, and therefore

$$b+ax-x^2 = -(x-m+in)(x-m-in) = -[(x-m)^2+n^2],$$

which is negative for all values of $x$. We shall consider only those cases where the factors are real.

### EXAMPLES

1. $$\int\frac{dx}{x\sqrt{x^2-x+2}}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{x^2-x+2}+x-\sqrt{2}}{\sqrt{x^2-x+2}+x+\sqrt{2}}+C.$$

2. $$\int\frac{dx}{x\sqrt{x^2+2x-1}}=2\arctan\left(x+\sqrt{x^2+2x-1}\right)+C.$$

3. $$\int\frac{dx}{\sqrt{2-x-x^2}}=-2\arctan\left(\frac{1-x}{x+2}\right)^{\frac{1}{2}}+C.$$

4. $$\int\frac{dx}{\sqrt{x^2-x-1}}=\log\left(2\sqrt{x^2-x+1}+2x-1\right)+C.$$

5. $$\int\frac{dx}{\sqrt{4x-3-x^2}}=-2\arctan\sqrt{\frac{1-x}{x-3}}+C.$$

6. $$\int\frac{xdx}{(2+3x-2x^2)^{\frac{3}{2}}}=\frac{8+6x}{25\sqrt{2+3x-2x^2}}+C.$$

7. $$\int\frac{5\,dx}{\sqrt{5x^2-2x+7}}=\sqrt{5}\log\left(5x-1+\sqrt{5}\sqrt{5x^2-2x+7}\right)+C.$$

8. $$\int\frac{dx}{\sqrt{3x^2-x+1}}=\frac{1}{\sqrt{3}}\log\left(6x-1+2\sqrt{3}\sqrt{3x^2-x+1}\right)+C.$$

9. $$\int\frac{4\,dx}{\sqrt{4+3x-2x^2}}=2\sqrt{2}\arcsin\left(\frac{4x-3}{\sqrt{41}}\right)+C.$$

10. $$\int\frac{dx}{\sqrt{x^2+x}}=\log\left(\frac{1}{2}+x+\sqrt{x^2+x}\right)+C.$$

11. $$\int\frac{(2x+x^2)^{\frac{1}{2}}dx}{x^2}=\log\left(x+1+\sqrt{2x+x^2}\right)-\frac{4}{x+\sqrt{2x+x^2}}+C.$$

The general integral treated in the last two sections has then the form

$$R(x,\ \sqrt{a+bx+cx^2})\,dx,$$

where $R$ denotes a rational function.

Combining the results of this chapter with the general theorem on page 328, we can then state the following

**Theorem.** *Every rational function of $x$ and the square root of a polynomial of degree not higher than the second can be integrated and the result expressed in terms of the elementary functions.**

---

* As before, however, it is assumed that in each case the denominator of the rational function can be broken up into real quadratic and linear factors.

**200. Binomial differentials.** A differential of the form

$$x^m(a+bx^n)^p\,dx,$$

where $a$ and $b$ are any constants and the exponents $m$, $n$, $p$ are rational numbers, is called a *binomial differential.*

Let $$x=z^\alpha;\text{ then } dx=\alpha z^{\alpha-1}dz,$$

and $$x^m(a+bx^n)^p\,dx=\alpha z^{m\alpha+\alpha-1}(a+bz^{n\alpha})^p\,dz.$$

If an integer $\alpha$ be chosen such that $m\alpha$ and $n\alpha$ are also integers,* we see that the given differential is equivalent to another of the same form where $m$ and $n$ have been replaced by integers. Also,

$$x^m(a+bx^n)^p\,dx=x^{m+np}(ax^{-n}+b)^p\,dx$$

transforms the given differential into another of the same form where the exponent $n$ of $x$ has been replaced by $-n$. Therefore, no matter what the algebraic sign of $n$ may be, in one of the two differentials the exponent of $x$ inside the parenthesis will surely be positive.

When $p$ is an integer the binomial may be expanded and the differential integrated termwise. In what follows $p$ is regarded as a fraction; hence we replace it by $\frac{r}{s}$, where $r$ and $s$ are integers.†

We may then make the following statement:

*Every binomial differential may be reduced to the form*

$$x^m(a+bx^n)^{\frac{r}{s}}\,dx,$$

*where $m$, $n$, $r$, $s$ are integers and $n$ is positive.*

**201. Conditions of integrability of the binomial differential**

$$(A)\qquad x^m(a+bx^n)^{\frac{r}{s}}\,dx.$$

CASE I. Assume $a+bx^n=z^s$.

Then $$(a+bx^n)^{\frac{1}{s}}=z,\text{ and } (a+bx^n)^{\frac{r}{s}}=z^r;$$

also $$x=\left(\frac{z^s-a}{b}\right)^{\frac{1}{n}},\text{ and } x^m=\left(\frac{z^s-a}{b}\right)^{\frac{m}{n}};$$

hence $$dx=\frac{s}{bn}z^{s-1}\left(\frac{z^s-a}{b}\right)^{\frac{1}{n}-1}dz.$$

---

* It is always possible to choose $\alpha$ so that $m\alpha$ and $n\alpha$ are integers, for we can take $\alpha$ as the L.C.M. of the denominators of $m$ and $n$.

† The case where $p$ is an integer is not excluded, but appears as a special case, viz., $r=p$, $s=1$.

Substituting in (*A*), we get

$$x^m(a+bx^n)^{\frac{r}{s}}dx = \frac{s}{bn}z^{r+s-1}\left(\frac{z^s-a}{b}\right)^{\frac{m+1}{n}-1}dz.$$

The second member of this expression is rational when

$$\frac{m+1}{n}$$

is an integer or zero.

Case II. Assume $a+bx^n = z^s x^n$.

Then $x^n = \frac{a}{z^s-b}$, and $a+bx^n = z^s x^n = \frac{az^s}{z^s-b}$.

Hence $(a+bx^n)^{\frac{r}{s}} = a^{\frac{r}{s}}(z^s-b)^{-\frac{r}{s}}z^r$;

also $x = a^{\frac{1}{n}}(z^s-b)^{-\frac{1}{n}}$, $x^m = a^{\frac{m}{n}}(z^s-b)^{-\frac{m}{n}}$;

and $dx = -\frac{s}{n}a^{\frac{1}{n}}z^{s-1}(z^s-b)^{-\frac{1}{n}-1}dz$.

Substituting in (*A*), we get

$$x^m(a+bx^n)^{\frac{r}{s}}dx = -\frac{s}{n}a^{\frac{m+1}{n}+\frac{r}{s}}(z^s-b)^{-\left(\frac{m+1}{n}+\frac{r}{s}+1\right)}z^{r+s-1}dz.$$

The second member of this expression is rational when $\frac{m+1}{n}+\frac{r}{s}$ is an integer or zero.

*Hence the binomial differential*

$$x^m(a+bx^n)^{\frac{r}{s}}dx$$

*can be integrated by rationalization in the following cases:**

**Case I.** *When* $\frac{m+1}{n} =$ *an integer or zero, by assuming*

$$a+bx^n = z^s.$$

**Case II.** *When* $\frac{m+1}{n}+\frac{r}{s} =$ *an integer or zero, by assuming*

$$a+bx^n = z^s x^n.$$

---

*Assuming as before that the denominator of the resulting rational function can be broken up into real quadratic and linear factors.

## EXAMPLES

1. $\displaystyle\int\frac{x^3dx}{(a+bx^2)^{\frac{3}{2}}}=\int x^3(a+bx^2)^{-\frac{3}{2}}dx=\frac{1}{b^2}\frac{2a+bx^2}{\sqrt{a+bx^2}}+C.$

*Solution.* $m=3$, $n=2$, $r=-3$, $s=2$; and here $\frac{m+1}{n}=2$, an integer. Hence this comes under Case I and we assume

$$a+bx^2=z^2;\text{ whence }x=\left(\frac{z^2-a}{b}\right)^{\frac{1}{2}},\ dx=\frac{zdz}{b^{\frac{1}{2}}(z^2-a)^{\frac{1}{2}}},\text{ and }(a+bx^2)^{\frac{3}{2}}=z^3.$$

$$\therefore\int\frac{x^3dx}{(a+bx^2)^{\frac{3}{2}}}=\int\left(\frac{z^2-a}{b}\right)^{\frac{3}{2}}\cdot\frac{zdz}{b^{\frac{1}{2}}(z^2-a)^{\frac{1}{2}}}\cdot\frac{1}{z^3}$$

$$=\frac{1}{b^2}\int(1-az^{-2})\,dz=\frac{1}{b^2}(z+az^{-1})+C$$

$$=\frac{1}{b^2}\frac{2a+bx^2}{\sqrt{a+bx^2}}+C.$$

2. $\displaystyle\int\frac{dx}{x^4\sqrt{1+x^2}}=\frac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}+C.$

*Solution.* $m=-4$, $n=2$, $\frac{r}{s}=-\frac{1}{2}$; and here $\frac{m+1}{n}+\frac{r}{s}=-2$, an integer. Hence this comes under Case II and we assume

$$1+x^2=z^2x^2,\ z=\frac{(1+x^2)^{\frac{1}{2}}}{x};$$

whence $\displaystyle x^2=\frac{1}{z^2-1},\ 1+x^2=\frac{z^2}{z^2-1},\ \sqrt{1+x^2}=\frac{z}{(z^2-1)^{\frac{1}{2}}};$

also $\displaystyle x=\frac{1}{(z^2-1)^{\frac{1}{2}}},\ x^4=\frac{1}{(z^2-1)^2};\text{ and }dx=-\frac{zdz}{(z^2-1)^{\frac{3}{2}}}.$

$$\therefore\int\frac{dx}{x^4\sqrt{1+x^2}}=-\int\frac{\dfrac{zdz}{(z^2-1)^{\frac{3}{2}}}}{\dfrac{1}{(z^2-1)^2}\cdot\dfrac{z}{(z^2-1)^{\frac{1}{2}}}}=-\int(z^2-1)\,dz$$

$$=z-\frac{z^3}{3}+C=\frac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}+C.$$

3. $\displaystyle\int x^3(1+x^2)^{\frac{1}{2}}dx=\frac{(3x^2-2)(1+x^2)^{\frac{3}{2}}}{15}+C.$

4. $\displaystyle\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}=\frac{x}{\sqrt{1+x^2}}+C.$

5. $\displaystyle\int\frac{x^3dx}{\sqrt{1+x^2}}=(1+x^2)^{\frac{1}{2}}\frac{(x^2-2)}{3}+C.$

6. $\displaystyle\int\frac{dx}{x\sqrt{a^2-x^2}}=\frac{1}{a}\log\frac{cx}{\sqrt{a^2-x^2}+a}.$

7. $\displaystyle\int\frac{adx}{x^2(1+x^2)^{\frac{3}{2}}}=-a(1+x^2)^{-\frac{1}{2}}\left(2x+\frac{1}{x}\right)+C.$

8. $\int \frac{dy}{y(a^2+y^2)^{\frac{1}{2}}} = \frac{1}{a}\log\frac{cy}{\sqrt{a^2+y^2}+a}.$

9. $\int t^3(1+2t^2)^{\frac{3}{2}}dt = (1+2t^2)^{\frac{5}{2}}\frac{5t^2-1}{70}+C.$

10. $\int u(1+u)^{\frac{3}{2}}du = \frac{2}{35}(1+u)^{\frac{5}{2}}(5u-2)+C.$

11. $\int \frac{\alpha^2 d\alpha}{(a+b\alpha^2)^{\frac{5}{2}}} = \frac{a}{3}\frac{\alpha^3}{(a+b\alpha^2)^{\frac{3}{2}}}+C.$

12. $\int \theta^5(1+\theta^2)^{\frac{2}{3}}d\theta = \frac{3}{22}(1+\theta^2)^{\frac{11}{3}} - \frac{3}{8}(1+\theta^2)^{\frac{8}{3}} + \frac{3}{10}(1+\theta^2)^{\frac{5}{3}}+C.$

13. $\int \frac{dx}{x^2(a+x^3)^{\frac{5}{3}}} = -\frac{3x^3+2a}{2a^2x(a+x^3)^{\frac{2}{3}}}+C.$

**202. Transformation of trigonometric differentials.**

From Trigonometry

(*A*) $$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}, \qquad 37, \text{ p. } 2$$

(*B*) $$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}. \qquad 37, \text{ p. } 2$$

But $$\sin\frac{x}{2} = \frac{1}{\csc\frac{x}{2}} = \frac{1}{\sqrt{\cot^2\frac{x}{2}+1}} = \frac{\tan\frac{x}{2}}{\sqrt{1+\tan^2\frac{x}{2}}},$$

and $$\cos\frac{x}{2} = \frac{1}{\sec\frac{x}{2}} = \frac{1}{\sqrt{1+\tan^2\frac{x}{2}}}.$$

If we now assume

$$\tan\frac{x}{2} = z, \text{ or, } x = 2 \text{ arc} \tan z,$$

we get $$\sin\frac{x}{2} = \frac{z}{\sqrt{1+z^2}}, \quad \cos\frac{x}{2} = \frac{1}{\sqrt{1+z^2}}.$$

Substituting in (*A*) and (*B*),

$$\sin x = \frac{2z}{1+z^2}, \quad \cos x = \frac{1-z^2}{1+z^2}.$$

Also by differentiating $x = 2$ arc tan $z$ we have $dx = \frac{2\,dz}{1+z^2}$.

Since $\sin x$, $\cos x$, and $dx$ are here expressed rationally in terms of $z$, it follows that

*A trigonometric differential involving $\sin x$ and $\cos x$ rationally only can be transformed by means of the substitution*

$$\tan\frac{x}{2} = z,$$

*or, what is the same thing, by the substitutions*

$$\sin x = \frac{2z}{1+z^2},\quad \cos x = \frac{1-z^2}{1+z^2},\quad dx = \frac{2\,dz}{1+z^2}$$

*into another differential expression which is rational in $z$.*

It is evident that if a trigonometric differential involves $\tan x$, $\cot x$, $\sec x$, $\csc x$ rationally only, it will be included in the above theorem, since these four functions can be expressed rationally in terms of $\sin x$, or $\cos x$, or both. It follows therefore that *any rational trigonometric differential can be integrated.*[*]

## EXAMPLES

1. $\displaystyle\int \frac{(1+\sin x)\,dx}{\sin x\,(1+\cos x)} = \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + \frac{1}{2}\log\tan\frac{x}{2} + C.$

*Solution.* Since this differential is rational in $\sin x$ and $\cos x$, we make the above substitutions at once, giving

$$\int \frac{(1+\sin x)\,dx}{\sin x\,(1+\cos x)} = \int \frac{\left(1+\dfrac{2z}{1+z^2}\right)\dfrac{2\,dz}{1+z^2}}{\dfrac{2z}{1+z^2}\left(1+\dfrac{1-z^2}{1+z^2}\right)}$$

$$= \int \frac{(1+z^2+2z)\,dz}{z\,(1+z^2+1-z^2)} = \frac{1}{2}\int (z+2+z^{-1})\,dz$$

$$= \frac{1}{2}\left(\frac{z^2}{2} + 2z + \log z\right) + C$$

$$= \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + \frac{1}{2}\log\left(\tan\frac{x}{2}\right) + C.$$

2. $\displaystyle\int \frac{dx}{4-5\sin x} = \frac{1}{3}\log\left(\frac{\tan\frac{x}{2}-2}{2\tan\frac{x}{2}-1}\right) + C.$

---

[*] See footnote, p. 335.

3. $\displaystyle\int\frac{dx}{5-3\cos x}=\frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right)+C.$

4. $\displaystyle\int\frac{dx}{5+4\sin 2x}=\frac{1}{3}\arctan\left(\frac{5\tan x+4}{3}\right)+C.$

5. $\displaystyle\int\frac{dx}{5-4\cos 2x}=\frac{1}{3}\arctan(3\tan x)+C.$

6. $\displaystyle\int\frac{dx}{3+5\cos x}=\frac{1}{4}\log\frac{\tan\frac{x}{2}+2}{\tan\frac{x}{2}-2}+C.$

7. $\displaystyle\int\frac{\sin x\,dx}{1+\sin x}=\frac{2}{1+\tan\frac{x}{2}}+2\arctan\left(\tan\frac{x}{2}\right)+C.$

8. $\displaystyle\int\frac{\cos x\,dx}{1+\cos x}=2\arctan\left(\tan\frac{x}{2}\right)-\tan\frac{x}{2}+C.$

9. Derive by the method of this article formulas [16] and [17], p. 293.

**203. Miscellaneous substitutions.** So far the substitutions considered have rationalized the given differential expression. In a great number of cases, however, integrations may be effected by means of substitutions which do not rationalize the given differential, but no general rule can be given, and the experience gained in working out a large number of problems must be our guide.

A very useful substitution is

$$x=\frac{1}{z},\quad dx=-\frac{dz}{z^2},$$

called the *reciprocal substitution*. Let us use this substitution in the next example.

Ex. 1. Find $\displaystyle\int\frac{\sqrt{a^2-x^2}}{x^4}dx.$

*Solution*. Making the substitution $x=\frac{1}{z}$, $dx=-\frac{dz}{z^2}$, we get

$$\int\frac{\sqrt{a^2-x^2}}{x^4}dx=-\int(a^2z^2-1)^{\frac{1}{2}}z\,dz=-\frac{(a^2z^2-1)^{\frac{3}{2}}}{3a^2}+C=-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C.$$

### EXAMPLES

1. $\displaystyle\int\frac{dx}{x(a^3+x^3)}=\frac{1}{3a^3}\log\frac{x^3}{a^3+x^3}+C.$ **Assume $x^3=z$.**

2. $\displaystyle\int\frac{x^2-x}{(x-2)^3}dx=\log(x-2)-\frac{3x-5}{(x-2)^2}+C.$ **Assume $x-2=z$.**

3. $\displaystyle\int\frac{x^3dx}{(x+1)^4}=\frac{18x^2+27x+11}{6(x+1)^3}+\log(x+1)+C.$ **Assume $x+1=z$.**

4. $\displaystyle\int\frac{dx}{(a^2+x^2)^{\frac{3}{2}}}=\frac{x}{a^2\sqrt{a^2+x^2}}+C.$ **Assume $x=\frac{1}{z}$.**

5. $\displaystyle\int\frac{dx}{x\sqrt{a^2+x^2}}=\frac{1}{a}\log\frac{cx}{a+\sqrt{a^2+x^2}}$ **Assume $x=\frac{a}{z}$.**

6. $\displaystyle\int\frac{x^3dx}{(x^2+1)^{\frac{2}{3}}}=\frac{3}{8}(x^2-3)(x^2+1)^{\frac{1}{3}}+C.$ **Assume $x^2+1=z$.**

7. $\displaystyle\int\frac{dx}{x\sqrt{1+x+x^2}}=\log\frac{cx}{2+x+2\sqrt{1+x+x^2}}.$ **Assume $x=\frac{1}{z}$.**

8. $\displaystyle\int\frac{\sqrt{1+\log x}}{x}dx=\frac{2}{3}(1+\log x)^{\frac{3}{2}}+C.$ **Assume $1+\log x=z$.**

9. $\displaystyle\int\frac{e^{2x}dx}{(e^x+1)^{\frac{1}{4}}}=\frac{4}{21}(3e^x-4)(e^x+1)^{\frac{3}{4}}+C.$ **Assume $e^x+1=z$.**

10. $\displaystyle\int\frac{dx}{e^{2x}-2e^x}=\frac{1}{2e^x}-\frac{x}{4}+\frac{1}{4}\log(e^x-2)+C.$ **Assume $e^x=z$.**

11. $\displaystyle\int\frac{x^2dx}{\sqrt{1-x^2}}=\frac{1}{2}\text{arc sin }x-\frac{x\sqrt{1-x^2}}{2}+C.$ **Assume $x=\cos z$.**

12. $\displaystyle\int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\text{arc sin}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C.$ **Assume $x=a\sin z$.**

# CHAPTER XXVIII

## INTEGRATION BY PARTS. REDUCTION FORMULAS

**204. Formula for integration by parts.** If $u$ and $v$ are functions of a single independent variable, we have from the formula for the differentiation of a product, V, p. 144,

$$d(uv) = udv + vdu,$$

or, transposing,

$$udv = d(uv) - vdu.$$

Integrating this, we get the inverse formula,

$$(A) \qquad \int udv = uv - \int vdu,$$

called the **formula for integration by parts.** This formula makes the integration of $udv$, which we may not be able to integrate directly, depend on the integration of $dv$ and $vdu$, which may be in such form as to be readily integrable. This method of *integration by parts* is one of the most useful in the Integral Calculus.

To apply this formula in any given case the given differential must be separated into two factors, namely, $u$ and $dv$. No general directions can be given for choosing these factors, except that

(*a*) *$dx$ is always a part of $dv$; and*

(*b*) *it must be possible to integrate $dv$.*

The following examples will show in détail how the formula is applied.

Ex. 1. Find $\int x \cos x dx$.

*Solution.* Let $u = x$ and $dv = \cos x dx$;

then $du = dx$ and $v = \int \cos x dx = \sin x$.

Substituting in $(A)$,

$$\int \overbrace{x}^{u} \overbrace{\cos x dx}^{dv} = \overbrace{x}^{u} \overbrace{\sin x}^{v} - \int \overbrace{\sin x}^{v} \overbrace{dx}^{du}$$

$$= x \sin x + \cos x + C.$$

Ex. 2. Find $\int x \log x dx$.

*Solution.* Let $u = \log x$ and $dv = xdx$;

then $du = \frac{dx}{x}$ and $v = \int xdx = \frac{x^2}{2}$.

Substituting in ($A$),

$$\int x \log x dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x}$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4} + C.$$

Ex. 3. Find $\int xe^{ax}dx$.

*Solution.* Let $u = e^{ax}$ and $dv = xdx$;

then $du = e^{ax} \cdot adx$ and $v = \int xdx = \frac{x^2}{2}$.

Substituting in ($A$),

$$\int xe^{ax}dx = e^{ax} \cdot \frac{x^2}{2} - \int \frac{x^2}{2} e^{ax}adx$$

$$= \frac{x^2e^{ax}}{2} - \frac{a}{2}\int x^2e^{ax}dx.$$

But $x^2e^{ax}dx$ is not as simple to integrate as $xe^{ax}dx$, which fact indicates that we did not choose our factors suitably. Instead

let $u = x$ and $dv = e^{ax}dx$;

then $du = dx$ and $v = \int e^{ax}dx = \frac{e^{ax}}{a}$.

Substituting in ($A$),

$$\int xe^{ax}dx = x \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} dx$$

$$= \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + C = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right) + C.$$

It may be necessary to apply the formula for integration by parts more than once, as in the following example.

Ex. 4. Find $\int x^2e^{ax}dx$.

*Solution.* Let $u = x^2$ and $dv = e^{ax}dx$;

then $du = 2\,xdx$ and $v = \int e^{ax}dx = \frac{e^{ax}}{a}$.

Substituting in ($A$),

$$\int x^2e^{ax}dx = x^2 \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot 2\,xdx$$

$$(B) \qquad = \frac{x^2e^{ax}}{a} - \frac{2}{a}\int xe^{ax}dx.$$

The integral in the last term may be found by applying formula (*A*) again, which gives

$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

Substituting this result in (*B*), we get

$$\int x^2e^{ax}dx = \frac{x^2e^{ax}}{a} - \frac{2\,e^{ax}}{a^2}\left(x - \frac{1}{a}\right) + C = \frac{e^{ax}}{a}\left(x^2 - \frac{2\,x}{a} + \frac{2}{a^2}\right) + C.$$

Among the most important applications of the method of integration by parts is the integration of

(*a*) *differentials involving products,*

(*b*) *differentials involving logarithms,*

(*c*) *differentials involving inverse circular functions.*

**EXAMPLES**

1. $\int x^2 \log x dx = \frac{x^3}{3}\left(\log x - \frac{1}{3}\right) + C.$

2. $\int \alpha \sin \alpha d\alpha = -\alpha \cos \alpha + \sin \alpha + C.$

3. $\int \text{arc} \sin x dx = x \,\text{arc} \sin x + \sqrt{1 - x^2} + C.$

*Hint.* Let $u = \text{arc} \sin x$ and $dv = dx$, etc.

4. $\int \log x dx = x(\log x - 1) + C.$

5. $\int \text{arc} \tan x dx = x \,\text{arc} \tan x - \log (1 + x^2)^{\frac{1}{2}} + C$

6. $\int x^n \log x dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C.$

7. $\int x \,\text{arc} \tan x dx = \frac{x^2 + 1}{2}\text{arc} \tan x - \frac{x}{2} + C.$

8. $\int \text{arc} \cot y dy = y \,\text{arc} \cot y + \frac{1}{2} \log (1 + y^2) + C.$

9. $\int xa^x dx = a^x\left[\frac{x}{\log a} - \frac{1}{\log^2 a}\right] + C.$

10. $\int t^2a^t dt = a^t\left[\frac{t^2}{\log a} - \frac{2\,t}{\log^2 a} + \frac{2}{\log^3 a}\right] + C.$

11. $\int \cos \theta \log \sin \theta d\theta = \sin \theta (\log \sin \theta - 1) + C.$

12. $\int x^2e^x dx = e^x(x^2 - 2\,x + 2) + C.$

13. $\int x^3 e^{ax} dx = \frac{e^{ax}}{a}\left(x^3 - \frac{3x^2}{a} + \frac{6x}{a^2} - \frac{6}{a^3}\right) + C.$

14. $\int \phi^2 \sin \phi d\phi = 2 \cos \phi + 2 \phi \sin \phi - \phi^2 \cos \phi + C.$

15. $\int (\log x)^2 dx = x [\log^2 x - 2 \log x + 2] + C.$

16. $\int \alpha \tan^2 \alpha d\alpha = \alpha \tan \alpha - \frac{\alpha^2}{2} + \log \cos \alpha + C.$

17. $\int \frac{\log x dx}{(x+1)^2} = \frac{x}{x+1} \log x - \log (x + 1) + C.$

*Hint.* Let $u = \log x$ and $dv = \frac{dx}{(x+1)^2}$, etc.

18. $\int x^2 \text{arc} \sin x dx = \frac{x^3}{3} \text{arc} \sin x + \frac{x^2+2}{9}\sqrt{1-x^2} + C.$

19. $\int \sec^2 \theta \log \tan \theta d\theta = \tan \theta (\log \tan \theta - 1) + C.$

20. $\int \log (\log x) \frac{dx}{x} = \log x \cdot \log (\log x) - \log x + C.$

21. $\int \frac{\log (x + 1) dx}{\sqrt{x+1}} = 2 \sqrt{x+1} [\log (x + 1) - 2] + C.$

22. $\int x^3 (a - x^2)^{\frac{1}{2}} dx = - \frac{1}{3} x^2 (a - x^2)^{\frac{3}{2}} - \frac{2}{15} (a - x^2)^{\frac{5}{2}} + C.$

*Hint.* Let $u = x^2$ and $dv = (a - x^2)^{\frac{1}{2}} x dx$, etc.

23. $\int \frac{(\log x)^2 dx}{x^{\frac{5}{2}}} = - \frac{2}{3 x^{\frac{3}{2}}} \left[\log^2 x + \frac{4}{3} \log x + \frac{8}{9}\right] + C.$

**205. Reduction formulas for binomial differentials.** It was shown in § 200, p. 334, that any binomial differential may be reduced to the form

$$x^m (a + bx^n)^p dx,$$

where $p$ is a rational number, $m$ and $n$ are integers, and $n$ is positive. Also in § 201, p. 334, we learned how to integrate such a differential expression in certain cases.

In general we can integrate such an expression by parts, using (*A*), p. 341, if it can be integrated at all. To apply the method of *integration by parts* to every example, however, is rather a long and tedious process. When the binomial differential cannot be integrated readily by any of the methods shown so far, it is customary to employ *reduction formulas* deduced by the method of integration by parts. By means of these reduction formulas the given differential is expressed as the sum of two terms, one of which is not affected by the sign of integration, and the other is an integral of the same form as the original expression, but one which is easier to integrate. The following are the four principal reduction formulas.

$$[A]\quad \int x^m(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{(np+m+1)b} - \frac{(m-n+1)a}{(np+m+1)b}\int x^{m-n}(a+bx^n)^p dx.$$

$$[B]\quad \int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$$

$$[C]\quad \int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^{p+1}}{(m+1)a} - \frac{(np+n+m+1)b}{(m+1)a}\int x^{m+n}(a+bx^n)^p dx.$$

$$[D]\quad \int x^m(a+bx^n)^p dx = -\frac{x^{m+1}(a+bx^n)^{p+1}}{n(p+1)a} + \frac{np+n+m+1}{n(p+1)a}\int x^m(a+bx^n)^{p+1}dx.$$

While it is not desirable for the student to memorize these formulas, he should know what each one will do, namely:

*Formula* [*A*] *diminishes* $m$ *by* $n$.
*Formula* [*B*] *diminishes* $p$ *by* 1.
*Formula* [*C*] *increases* $m$ *by* $n$.
*Formula* [*D*] *increases* $p$ *by* 1.

I. *To derive formula* [*A*].

The formula for integration by parts is

$$(A)\qquad \int u\,dv = uv - \int v\,du. \qquad (A),\ \text{p. } 341$$

We may apply this formula in the integration of

$$\int x^m(a+bx^n)^p dx$$

by placing $u = x^{m-n+1}$* and $dv = (a+bx^n)^p x^{n-1}dx$;

then $du = (m-n+1)x^{m-n}$ and $v = \dfrac{(a+bx^n)^{p+1}}{nb(p+1)}$.

---

* In order to integrate $dv$ by [4] it is necessary that $x$ outside the parenthesis shall have the exponent $n-1$. Subtracting $n-1$ from $m$ leaves $m-n+1$ for the exponent of $x$ in $u$.

Substituting in $(A)$,

$$(B)\quad \int x^m(a+bx^n)^p\,dx$$
$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}-\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx.$$

But $\int x^{m-n}(a+bx^n)^{p+1}dx=\int x^{m-n}(a+bx^n)^p(a+bx^n)\,dx$

$$=a\int x^{m-n}(a+bx^n)^p\,dx+b\int x^m(a+bx^n)^p\,dx.$$

Substituting this in $(B)$, we get

$$\int x^m(a+bx^n)^p\,dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}$$
$$-\frac{(m-n+1)a}{nb(p+1)}\int x^{m-n}(a+bx^n)^p\,dx-\frac{m-n+1}{n(p+1)}\int x^m(a+bx^n)^p\,dx.$$

Transposing the last term to the first member, combining, and solving for $\int x^m(a+bx^n)^p\,dx$, we obtain

$$[A]\quad \int \boldsymbol{x^m(a+bx^n)^p\,dx}$$
$$=\boldsymbol{\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)}-\frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p\,dx.}$$

It is seen by formula $[A]$ that the integration of $x^m(a+bx^n)^p\,dx$ is made to depend upon the integration of another differential of the same form in which $m$ is replaced by $m-n$. By repeated applications of formula $[A]$, $m$ may be diminished by any multiple of $n$.

When $np+m+1=0$, formula $[A]$ evidently fails (the denominator vanishing). But in that case

$$\frac{m+1}{n}+p=0;$$

hence we can apply the method of § 201, p. 335, and the formula is not needed.

II. *To derive formula* $[B]$. Separating the factors, we may write

$$(C)\quad \int x^m(a+bx^n)^p dx=\int x^m(a+bx^n)^{p-1}(a+bx^n)\,dx$$
$$=a\int x^m(a+bx^n)^{p-1}dx+b\int x^{m+n}(a+bx^n)^{p-1}dx.$$

Now let us apply formula [*A*] to the last term of (*C*) by substituting in the formula $m+n$ for $m$ and $p-1$ for $p$. This gives

$$b\int x^{m+n}(a+bx^n)^{p-1}dx = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} - \frac{a(m+1)}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$$

Substituting this in (*C*), and combining like terms, we get

$$[B]\quad \int x^m(a+bx^n)^p dx$$

$$= \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$$

Each application of formula [*B*] diminishes $p$ by unity. Formula [*B*] fails for the same case as [*A*].

III. *To derive formula* [*C*]. Solving formula [*A*] for

$$\int x^{m-n}(a+bx^n)^p dx,$$

and substituting $m+n$ for $m$, we get

$$[C]\quad \int x^m(a+bx^n)^p dx$$

$$= \frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)} - \frac{b(np+n+m+1)}{a(m+1)}\int x^{m+n}(a+bx^n)^p dx.$$

Therefore each time we apply [*C*], $m$ is replaced by $m+n$. When $m+1=0$, formula [*C*] fails, but then the differential expression can be integrated by the method of § 201, p. 335, and the formula is not needed.

IV. *To derive formula* [*D*]. Solving formula [*B*] for

$$\int x^m(a+bx^n)^{p-1}dx,$$

and substituting $p+1$ for $p$, we get

$$[D]\quad \int x^m(a+bx^n)^p dx$$

$$= -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)} + \frac{np+n+m+1}{an(p+1)}\int x^m(a+bx^n)^{p+1}dx.$$

Each application of [*D*] increases $p$ by unity. Evidently [*D*] fails when $p+1=0$, but then $p=-1$ and the expression is rational.

## EXAMPLES

1. $\int \frac{x^3dx}{\sqrt{1-x^2}} = -\frac{1}{3}(x^2+2)(1-x^2)^{\frac{1}{2}} + C.$

*Solution.* Here $m = 3$, $n = 2$, $p = -\frac{1}{2}$, $a = 1$, $b = -1$.

We apply reduction formula [$A$] in this case because the integration of the differential would then depend on the integration of $\int x(1-x^2)^{-\frac{1}{2}}dx$, which comes under [4], p. 292. Hence, substituting in [$A$], we obtain

$$\begin{aligned}\int x^3(1-x^2)^{-\frac{1}{2}}dx &= \frac{x^{3-2+1}(1-x^2)^{-\frac{1}{2}+1}}{-1(-1+3+1)} - \frac{1(3-2+1)}{-1(-1+3+1)}\int x^{3-2}(1-x^2)^{-\frac{1}{2}}dx\\ &= -\tfrac{1}{3}x^2(1-x^2)^{\frac{1}{2}} + \tfrac{2}{3}\int x(1-x^2)^{-\frac{1}{2}}dx\\ &= -\tfrac{1}{3}x^2(1-x^2)^{\frac{1}{2}} - \tfrac{2}{3}(1-x^2)^{\frac{1}{2}} + C\\ &= -\tfrac{1}{3}(x^2+2)\cdot(1-x^2)^{\frac{1}{2}} + C.\end{aligned}$$

2. $\int \frac{x^4dx}{(a^2-x^2)^{\frac{1}{2}}} = -\left(\frac{1}{4}x^3 + \frac{3}{8}a^2x\right)\sqrt{a^2-x^2} + \frac{3}{8}a^4 \operatorname{arc}\sin\frac{x}{a} + C.$

*Hint.* Apply [$A$] twice.

3. $\int (a^2+x^2)^{\frac{1}{2}}dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log(x+\sqrt{a^2+x^2}) + C.$

*Hint.* Here $m = 0$, $n = 2$, $p = \frac{1}{2}$, $a = a^2$, $b = 1$. Apply [$B$] once.

4. $\int \frac{dx}{x^3\sqrt{x^2-1}} = \frac{(x^2-1)^{\frac{1}{2}}}{2x^2} + \frac{1}{2}\operatorname{arc}\sec x + C.$

*Hint.* Apply [$C$] once.

5. $\int \frac{x^2dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\operatorname{arc}\sin\frac{x}{a} + C.$

6. $\int \frac{x^3dx}{\sqrt{a^2+x^2}} = \frac{1}{3}(x^2-2a^2)\sqrt{a^2+x^2} + C.$

7. $\int \frac{x^5dx}{\sqrt{1-x^2}} = -\left(\frac{x^4}{5} + \frac{4x^2}{15} + \frac{8}{15}\right)\sqrt{1-x^2} + C.$

8. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\operatorname{arc}\sin\frac{x}{a} + C.$

*Hint.* Apply [$A$] and then [$B$].

9. $\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\operatorname{arc}\tan\frac{x}{a} + C.$

*Hint.* Apply [$D$] once.

10. $\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{x}{a+\sqrt{a^2-x^2}} + C.$

11. $\int \frac{x^3dx}{(a^2+x^2)^{\frac{3}{2}}} = \frac{x^2+2a^2}{(a^2+x^2)^{\frac{1}{2}}} + C.$

12. $\int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}} = \frac{(3a^2-2x^2)x}{3a^4(a^2-x^2)^{\frac{3}{2}}} + C.$

13. $\int (x^2+a^2)^{\frac{3}{2}}dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^4}{8}\log(x+\sqrt{x^2+a^2}) + C.$

14. $\int x^2(x^2+a^2)^{\frac{1}{2}}dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2+a^2}) + C.$

15. $\int \frac{x^2dx}{\sqrt{2ax-x^2}} = -\frac{x+3a}{2}(2ax-x^2)^{\frac{1}{2}} + \frac{3a^2}{2}\text{arc vers}\frac{x}{a} + C.$

*Hint.* $\int \frac{x^2dx}{\sqrt{2ax-x^2}} = \int x^{\frac{3}{2}}(2a-x)^{-\frac{1}{2}}dx.$ Apply [*A*] twice.

16. $\int \frac{dx}{x^2(a^2-x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C.$

17. $\int \frac{y^3dy}{\sqrt{2ry-y^2}} = -\frac{2y^2+5r(y+3r)}{6}\sqrt{2ry-y^2} + \frac{5}{2}r^3\text{arc vers}\frac{y}{r} + C.$

18. $\int \frac{tdt}{\sqrt{2at-t^2}} = -(2at-t^2)^{\frac{1}{2}} + a\,\text{arc vers}\frac{t}{a} + C.$

19. $\int \frac{ds}{(a^2+s^2)^3} = \frac{s}{4a^2(a^2+s^2)^2} + \frac{3s}{8a^4(a^2+s^2)} + \frac{3}{8a^5}\text{arc tan}\frac{s}{a} + C.$

20. $\int \frac{r^8dr}{\sqrt{1-r^3}} = -\frac{2}{45}(3r^6+4r^3+8)\sqrt{1-r^3} + C.$

**206. Reduction formulas for trigonometric differentials.** The method of the last section, which makes the given integral depend on another integral of the same form, is called *successive reduction.*

We shall now apply the same method to trigonometric differentials by deriving and illustrating the use of the following **trigonometric reduction formulas.**

[*E*] $\int \sin^m x\cos^n x\,dx$

$$= \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx.$$

[*F*] $\int \sin^m x\cos^n x\,dx$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx.$$

$$[G]\quad \int \sin^m x \cos^n x\, dx = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x\, dx.$$

$$[H]\quad \int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x\, dx.$$

Here the student should note that

*Formula* [*E*] *diminishes* $n$ *by* 2.
*Formula* [*F*] *diminishes* $m$ *by* 2.
*Formula* [*G*] *increases* $n$ *by* 2.
*Formula* [*H*] *increases* $m$ *by* 2.

To derive these we apply as before the formula for integration by parts, namely,

$$(A)\qquad \int u\,dv = uv - \int v\,du. \qquad (A),\ \text{p. } 341$$

Let $u = \cos^{n-1} x$, and $dv = \sin^m x \cos x\, dx$;

then $du = -(n-1)\cos^{n-2} x \sin x\, dx$, and $v = \dfrac{\sin^{m+1} x}{m+1}$.

Substituting in (*A*), we get

$$(B)\qquad \int \sin^m x \cos^n x\, dx = +\frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\, dx.$$

In the same way, if we
let $u = \sin^{m-1} x$, and $dv = \cos^n x \sin x\, dx$,
we obtain

$$(C)\qquad \int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x\, dx.$$

But
$$\int \sin^{m+2} x \cos^{n-2} x\, dx = \int \sin^m x (1 - \cos^2 x) \cos^{n-2} x\, dx$$
$$= \int \sin^m x \cos^{n-2} x\, dx - \int \sin^m x \cos^n x\, dx.$$

Substituting this in (*B*), combining like terms, and solving for $\int \sin^m x \cos^n x\,dx$, we get

$$[E]\quad \int \sin^m x \cos^n x\,dx$$
$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\,dx.$$

Making a similar substitution in (*C*), we get

$$[F]\quad \int \sin^m x \cos^n x\,dx$$
$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\,dx.$$

Solving formula [*E*] for the integral on the right-hand side, and increasing $n$ by 2, we get

$$[G]\quad \int \sin^m x \cos^n x\,dx$$
$$= -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x\,dx.$$

In the same way we get from formula [*F*],

$$[H]\quad \int \sin^m x \cos^n x\,dx$$
$$= \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x\,dx.$$

Formulas [*E*] and [*F*] fail when $m+n=0$, formula [*G*] when $n+1=0$, and formula [*H*] when $m+1=0$. But in such cases we may integrate by methods which have been previously explained.

It is clear that when $m$ and $n$ are integers the integral

$$\int \sin^m x \cos^n x\,dx$$

may be made to depend, by using one of the above reduction formulas, upon one of the following integrals:

$$\int dx,\ \int \sin x\,dx,\ \int \cos x\,dx,\ \int \sin x \cos x\,dx,\ \int \frac{dx}{\sin x},\ \int \frac{dx}{\cos x},$$
$$\int \frac{dx}{\cos x \sin x},\ \int \tan x\,dx,\ \int \cot x\,dx,$$

all of which we have learned how to integrate.

**EXAMPLES**

1. $\int \sin^2 x \cos^4 x dx = -\frac{\sin x \cos^5 x}{6} + \frac{\sin x \cos^3 x}{24} + \frac{1}{16}(\sin x \cos x + x) + C.$

*Solution.* First applying formula [$F$], we get

$$(A)\qquad \int \sin^2 x \cos^4 x dx = -\frac{\sin x \cos^5 x}{6} + \frac{1}{6}\int \cos^4 x dx.$$

[Here $m = 2$, $n = 4$.]

Applying formula [$E$] to the integral in the second member of ($A$), we get

$$(B)\qquad \int \cos^4 x dx = \frac{\sin x \cos^3 x}{6} + \frac{3}{4}\int \cos^2 x dx.$$

[Here $m = 0$, $n = 4$.]

Applying formula [$E$] to the second member of ($B$) gives

$$(C)\qquad \int \cos^2 x dx = \frac{\sin x \cos x}{2} + \frac{x}{2}.$$

Now substitute the result ($C$) in ($B$), and then this result in ($A$). This gives the answer as above.

2. $\int \sin^4 x \cos^2 x dx = \frac{\cos x}{2}\left(\frac{\sin^5 x}{3} - \frac{\sin^3 x}{12} - \frac{\sin x}{8}\right) + \frac{x}{16} + C.$

3. $\int \frac{dx}{\sin^4 x \cos^2 x} = \tan x - 2 \cot x - \frac{1}{3}\cot^3 x + C.$

4. $\int \frac{\cos^4 x dx}{\sin^2 x} = -\frac{\cot x}{2}(3 - \cos^2 x) - \frac{3x}{2} + C.$

5. $\int \frac{\cos^4 a da}{\sin^3 a} = -\frac{\cos a}{2 \sin^2 a} - \cos a - \frac{3}{2}\log \tan \frac{a}{2} + C.$

6. $\int \sin^6 a da = -\frac{\cos a}{2}\left(\frac{\sin^5 a}{3} + \frac{5}{12}\sin^3 a + \frac{5}{8}\sin a\right) + \frac{5a}{16} + C.$

7. $\int \frac{d\theta}{\sin^5 \theta} = -\frac{\cos \theta}{4}\left(\frac{1}{\sin^4 \theta} + \frac{3}{2 \sin^2 \theta}\right) + \frac{3}{8}\log \tan \frac{\theta}{2} + C.$

8. $\int \frac{d\phi}{\cos^7 \phi} = \frac{\sin \phi}{2 \cos^2 \phi}\left(\frac{1}{3 \cos^4 \phi} + \frac{5}{12 \cos^2 \phi} + \frac{5}{8}\right) + \frac{5}{16}\log(\sec \phi \tan \phi) + C.$

9. $\int \cos^8 t dt = \frac{\sin t}{8}\left(\cos^7 t + \frac{7}{6}\cos^5 t + \frac{35}{24}\cos^3 t + \frac{35}{16}\cos t\right) + \frac{35 t}{128} + C.$

10. $\int \frac{dy}{\sin^4 y \cos^3 y} = -\frac{1}{\cos^2 y}\left(\frac{1}{3 \sin^3 y} + \frac{5}{3 \sin y} - \frac{5}{2}\sin y\right)$
$+ \frac{5}{2}\log(\sec y + \tan y) + C.$

**207.** To find $\int e^{ax}\sin nx\,dx$ and $\int e^{ax}\cos nx\,dx$.

Integrating $e^{ax}\sin nx\,dx$ by parts,

letting $$u = e^{ax}, \text{ and } dv = \sin nx\,dx;$$

then $$du = ae^{ax}dx, \text{ and } v = -\frac{\cos nx}{n}.$$

Substituting in formula (*A*), p. 341, namely,

$$\int u\,dv = uv - \int v\,du,$$

we get

$$(A) \qquad \int e^{ax}\sin nx\,dx = -\frac{e^{ax}\cos nx}{n} + \frac{a}{n}\int e^{ax}\cos nx\,dx$$

Integrating $e^{ax}\sin nx\,dx$ again by parts,

letting $$u = \sin nx, \text{ and } dv = e^{ax}dx;$$

then $$du = n\cos nx\,dx, \text{ and } v = \frac{e^{ax}}{a}.$$

Substituting in (*A*), p. 341, we get

$$(B) \qquad \int e^{ax}\sin nx\,dx = \frac{e^{ax}\sin nx}{a} - \frac{n}{a}\int e^{ax}\cos nx\,dx.$$

Eliminating $\int e^{ax}\cos nx\,dx$ between (*A*) and (*B*), we have

$$(a^2 + n^2)\int e^{ax}\sin nx\,dx = e^{ax}(a\sin nx - n\cos nx),$$

or, $$\int e^{ax}\sin nx\,dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2 + n^2} + C.$$

Similarly we may obtain

$$\int e^{ax}\cos nx\,dx = \frac{e^{ax}(n\sin nx + a\cos nx)}{a^2 + n^2} + C.$$

In working out the examples which follow, the student is advised not to use the above results as formulas, but to follow the method by which they were obtained.

**EXAMPLES**

1. $\int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C.$

2. $\int e^x \cos x dx = \frac{e^x}{2}(\sin x + \cos x) + C.$

3. $\int e^{2x} \cos 3x dx = \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C.$

4. $\int \frac{\sin x dx}{e^x} = -\frac{\sin x + \cos x}{2e^x} + C.$

5. $\int \frac{\cos 2x dx}{e^{3x}} = \frac{1}{13e^{3x}}(2 \sin 2x - 3 \cos 2x) + C.$

6. $\int e^x \sin^2 x dx = \frac{e^x}{2}\left(1 - \frac{2 \sin 2x + \cos 2x}{5}\right) + C.$

7. $\int e^\alpha \cos^2 \alpha d\alpha = \frac{e^\alpha}{2}\left(1 + \frac{2 \sin 2\alpha + \cos 2\alpha}{5}\right) + C.$

8. $\int e^{\frac{x}{2}} \cos \frac{x}{2} dx = e^{\frac{x}{2}}\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) + C.$

9. $\int e^{a\alpha}(\sin a\alpha + \cos a\alpha) d\alpha = \frac{e^{a\alpha} \sin a\alpha}{a} + C.$

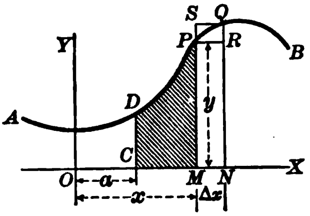
10. $\int e^{3x}(\sin 2x - \cos 2x) dx = \frac{e^{3x}}{13}(\sin 2x - 5 \cos 2x) + C.$

# CHAPTER XXIX

## THE DEFINITE INTEGRAL

**208. Differential of an area.** Consider the continuous function $\phi(x)$, and let

$$y = \phi(x)$$



be the equation of the curve $AB$. Let $CD$ be a fixed and $MP$ a variable ordinate, and let $u$ be the measure of the area $CMPD$.* When $x$ takes on a sufficiently small increment $\Delta x$, $u$ takes on an increment $\Delta u$ (= area $MNQP$). Completing the rectangles $MNRP$ and $MNQS$, we see that

$$\text{area } MNRP < \text{area } MNQP < \text{area } MNQS,$$

or,

$$MP \cdot \Delta x < \Delta u < NQ \cdot \Delta x;$$

and, dividing by $\Delta x$,

$$MP < \frac{\Delta u}{\Delta x} < NQ.\dagger$$

Now let $\Delta x$ approach zero as a limit; then since $MP$ remains fixed and $NQ$ approaches $MP$ as a limit (since $y$ is a continuous function of $x$), we get

$$\frac{du}{dx} = y (= MP),$$

or, using differentials,

$$du = y dx.$$

**Theorem.** *The differential of the area bounded by any curve, the axis of X, and two ordinates is equal to the product of the ordinate terminating the area and the differential of the corresponding abscissa.*

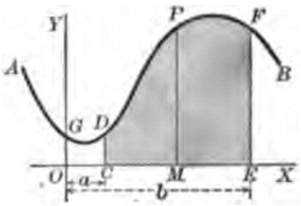
---

* We may suppose this area to be generated by a variable ordinate starting out from $CD$ and moving to the right; hence $u$ will be a function of $x$ which vanishes when $x = a$.

† In this figure $MP$ is less than $NQ$; if $MP$ happens to be greater than $NQ$, simply reverse the inequality signs.

**209. The definite integral.** It follows from the theorem in the last section that if $AB$ is the locus of

$$y = \phi(x),$$



then $$du = y\,dx, \text{ or,}$$

(A) $$du = \phi(x)\,dx,$$

where $du$ is the differential of the area between the curve, the axis of $x$, and any two ordinates. Integrating (A), we get

$$u = \int \phi(x)\,dx.$$

Since $\int \phi(x)\,dx$ exists (it is here represented geometrically as an area), denote it by $f(x) + C$.

(B) $$\therefore u = f(x) + C.$$

We may determine $C$, as in Chapter XXV, if we know the value of $u$ for some value of $x$. If we agree to reckon the area from the axis of $y$, i.e. when

(C) $x = a$, $u =$ area $OCDG$,

and when $x = b$, $u =$ area $OEFG$, etc.,

it follows that if

(D) $x = 0$, then $u = 0$.

Substituting (D) in (B), we get

$$u = f(0) + C, \text{ or } C = -f(0).$$

Hence from (B) we obtain

(E) $$u = f(x) - f(0),$$

giving the area from the axis of $y$ to any ordinate (as $MP$).

To find the area between the ordinates $CD$ and $EF$, substitute the values (C) in (E), giving

(F) area $OCDG = f(a) - f(0)$,

(G) area $OEFG = f(b) - f(0)$.

Subtracting (F) from (G),

(H) **area $CEFD = f(b) - f(a)$.**[*]

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[*] The student should observe that under the present hypothesis $f(x)$ will be a *single-valued* function which changes *continuously* from $f(a)$ to $f(b)$ as $x$ changes from $a$ to $b$.

**Theorem.** *The difference of the values of $\int y\,dx$ for $x = a$ and $x = b$ gives the area bounded by the curve whose ordinate is $y$, the axis of $X$, and the ordinates corresponding to $x = a$ and $x = b$.*

This difference is represented by the symbol*

$$(I) \qquad \int_a^b y\,dx, \text{ or } \int_a^b \phi(x)\,dx,$$

and is read "the integral from $a$ to $b$ of $y\,dx$." The operation is called *integration between limits*, $a$ being the *lower* and $b$ the *upper* limit.†

Since $(I)$ always has a *definite* value, it is called a *definite integral*. For, if

$$\int \phi(x)\,dx = f(x) + C,$$

then $$\int_a^b \phi(x)\,dx = \Big[f(x) + C\Big]_a^b = \Big[f(b) + C\Big] - \Big[f(a) + C\Big],$$

or, $$\int_a^b \phi(x)\,dx = f(b) - f(a),$$

the *constant of integration* having disappeared.

*We may accordingly define the symbol*

$$\int_a^b \phi(x)\,dx$$

*as the numerical measure of the area bounded by the curve $y = \phi(x)$,‡ the axis of $X$, and the ordinates of the curve at $x = a$, $x = b$. This definition presupposes that these lines bound an area, i.e. the curve does not rise or fall to infinity, and both $a$ and $b$ are finite.*

We have shown that the numerical value of the definite integral is always $f(b) - f(a)$, but we shall see in Ex. 2, p. 363, that $f(b) - f(a)$ may be a number when the definite integral has no meaning.

**210. Geometrical representation of an integral.** In the last section we represented the definite integral as an area. This does not necessarily mean that every integral *is* an area, for the physical interpretation of the result depends on the nature of the quantities represented by the abscissa and the ordinate. Thus, if $x$ and $y$ are considered as simply the coördinates of a point and nothing more, then the integral is indeed an area. But suppose the ordinate represents the speed of a moving point, and the corresponding abscissa the time at which the point has that speed; then the graph is the speed curve of the motion, and the area under it and between any two ordinates will *represent* the distance passed through in the corresponding interval of time (see § 187). That is, the *number* which denotes the area equals the *number* which denotes the distance (or value of the integral).

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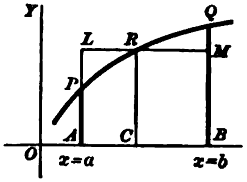
* This notation is due to Joseph Fourier (1768–1830).

† The word *limit* in this connection means merely the value of the variable at one end of its range (end value), and should not be confused with the meaning of the word in the Theory of Limits.

‡ $\phi(x)$ is continuous and single-valued throughout the interval $[a, b]$.

Similarly a definite integral standing for volume, surface, mass, force, etc., may be represented geometrically by an area. On page 372 the algebraic sign of an area is interpreted.

**211. Mean value of $\phi(x)$.** This is defined as follows:



$$\left.\begin{array}{l}\textbf{Mean value of } \phi(x)\\ \textbf{from } x=a \textbf{ to } x=b\end{array}\right\}=\frac{\int_a^b \phi(x)\,dx}{b-a}.$$

Since from the figure

$$\int_a^b \phi(x)\,dx = \text{area } APQB,$$

this means that if we construct on the base $AB(=b-a)$ a rectangle (as $ALMB$) whose area equals the area of $APQB$, then

$$\text{mean value} = \frac{\text{area } ALMB}{b-a} = \frac{AB\cdot CR}{AB} = \text{altitude } CR.$$

**212. Interchange of limits.**

Since $\int_a^b \phi(x)\,dx = f(b) - f(a)$, and

$$\int_b^a \phi(x)\,dx = f(a) - f(b) = -[f(b) - f(a)],$$

we have

$$\int_a^b \phi(x)\,dx = -\int_b^a \phi(x)\,dx.$$

**Theorem.** *Interchanging the limits is equivalent to changing the sign of the definite integral.*

**213. Decomposition of the interval of integration of the definite integral.**

Since $$\int_a^{x_1}\phi(x)\,dx = f(x_1) - f(a) \text{ and}$$

$$\int_{x_1}^{b}\phi(x)\,dx = f(b) - f(x_1),$$

we get by addition,

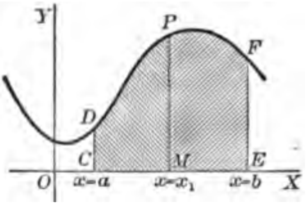
$$\int_a^{x_1}\phi(x)\,dx + \int_{x_1}^{b}\phi(x)\,dx = f(b) - f(a).$$

But $$\int_a^{b}\phi(x)\,dx = f(b) - f(a);$$

therefore by comparing the last two expressions we obtain

$$\int_a^{b}\phi(x)\,dx = \int_a^{x_1}\phi(x)\,dx + \int_{x_1}^{b}\phi(x)\,dx.$$

Interpreting this theorem geometrically, as in § 209, p. 356, we see that the integral on the left-hand side represents the whole area *CEFD*, the first integral on the right-hand side the area *CMPD*, and the second integral on the right-hand side the area *MEFP*. The truth of the theorem is therefore obvious.



Even if $x_1$ does not lie in the interval between $a$ and $b$, the truth of the theorem is apparent when the sign (p. 372) as well as the magnitude of the areas is taken into account. Evidently the definite integral may be decomposed into any number of separate definite integrals in this way.

**214. The definite integral a function of its limits.**

From $$\int_a^{b}\phi(x)\,dx = f(b) - f(a)$$

we see that the definite integral is a function of its limits. Thus $\int_a^{b}\phi(z)\,dz$ has precisely the same value as $\int_a^{b}\phi(x)\,dx$.

**Theorem.** *A definite integral is a function of its limits.*

**215. Calculation of a definite integral.** The process may be summarized as follows:

**First step.** *Find the indefinite integral of the given differential expression.*

**Second step.** *Substitute in this indefinite integral first the upper limit and then the lower limit for the variable, and subtract the last result from the first.*

It is not necessary to bring in the constant of integration, since it always disappears in subtracting.

**Ex. 1.** Find $\int_1^4 x^2dx$.

*Solution.* $\int_1^4 x^2dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = 21.$ *Ans.*

**Ex. 2.** Find $\int_0^\pi \sin x dx$.

*Solution.* $\int_0^\pi \sin x dx = \Big[-\cos x\Big]_0^\pi = \Big[-(-1)\Big] - \Big[-1\Big] = 2.$ *Ans.*

**Ex. 3.** Find $\int_0^a \frac{dx}{a^2 + x^2}$.

*Solution.*

$$\int_0^a \frac{dx}{a^2+x^2} = \left[\frac{1}{a}\arctan\frac{x}{a}\right]_0^a = \frac{1}{a}\arctan 1 - \frac{1}{a}\arctan 0$$

$$= \frac{\pi}{4a} - 0 = \frac{\pi}{4a}. \quad \textit{Ans.}$$

**216. Infinite limits.** So far the limits of the integral have been assumed as finite. Even in elementary work, however, it is sometimes desirable to remove this restriction and to consider integrals with infinite limits. This is possible in certain cases by making use of the following *definitions.*

When the upper limit is infinite,

$$\int_a^{+\infty} \phi(x)\,dx = \underset{b=+\infty}{\text{limit}} \int_a^b \phi(x)\,dx,$$

and when the lower limit is infinite,

$$\int_{-\infty}^b \phi(x)\,dx = \underset{a=-\infty}{\text{limit}} \int_a^b \phi(x)\,dx,$$

provided the limits exist.

**Ex. 1.** Find $\int_1^{+\infty}\frac{dx}{x^2}$.

*Solution.* $\int_1^{\infty}\frac{dx}{x^2}=\underset{b=+\infty}{\text{limit}}\int_1^b\frac{dx}{x^2}=\underset{b=+\infty}{\text{limit}}\left[-\frac{1}{x}\right]_1^b$

$$=\underset{b=+\infty}{\text{limit}}\left[-\frac{1}{b}+1\right]=1.\quad \textit{Ans.}$$

**Ex. 2.** Find $\int_0^{+\infty}\frac{8a^3dx}{x^2+4a^2}$.

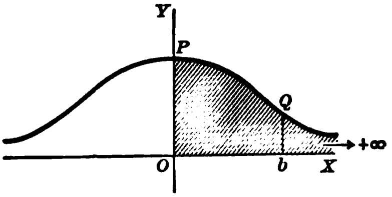
*Solution.* $\int_0^{+\infty}\frac{8a^3dx}{x^2+4a^2}=\underset{b=+\infty}{\text{limit}}\int_0^b\frac{8a^3dx}{x^2+4a^2}=\underset{b=+\infty}{\text{limit}}\left[4a^2\arc\tan\frac{x}{2a}\right]_0^b$

$$=\underset{b=+\infty}{\text{limit}}\left[4a^2\text{arc}\tan\frac{b}{2a}\right]=4a^2\cdot\frac{\pi}{2}=2\pi a^2.\quad \textit{Ans.}$$

Let us interpret this result geometrically. The graph of our function is the witch, the locus of

$$y=\frac{8a^3}{x^2+4a^2}.$$

$$\text{Area } OPQb=\int_0^b\frac{8a^3dx}{x^2+4a^2}=4a^2\,\text{arc}\tan\frac{b}{2a}.$$



Now as the ordinate $Qb$ moves indefinitely to the right,

$$4a^2\,\text{arc}\tan\frac{b}{2a}$$

is always finite, and

$$\underset{b=+\infty}{\text{limit}}\left[4a^2\,\text{arc}\tan\frac{b}{2a}\right]=2\pi a^2,$$

which is also finite. In such cases *we call the result the area bounded by the curve, the ordinate OP, and OX*, although strictly speaking this area is not completely bounded.

**Ex. 3.** Find $\int_1^{+\infty}\frac{dx}{x}$.

*Solution.* $$\int_1^{+\infty}\frac{dx}{x}=\underset{b=+\infty}{\text{limit}}\int_1^b\frac{dx}{x}=\underset{b=+\infty}{\text{limit}}(\log b).$$

The limit of $\log b$ as $b$ increases without limit does not exist; hence the integral has in this case no meaning.

**217. When $\phi(x)$ is discontinuous.** Let us now consider cases when the function to be integrated is discontinuous for isolated values of the variable lying within the limits of integration.

Consider first the case where the function to be integrated is continuous for all values of $x$ between the limits $a$ and $b$ except $x=a$.

If $a < b$ and $\epsilon$ is positive, we use the *definition*

$$(A)\qquad \int_a^b \phi(x)\,dx = \underset{\epsilon=0}{\text{limit}} \int_{a+\epsilon}^b \phi(x)\,dx,$$

and when $\phi(x)$ is continuous except at $x = b$, we use the *definition*

$$(B)\qquad \int_a^b \phi(x)\,dx = \underset{\epsilon=0}{\text{limit}} \int_a^{b-\epsilon} \phi(x)\,dx,$$

provided the limits are definite quantities.

Ex. 1. Find $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$.

*Solution.* Here $\frac{1}{\sqrt{a^2 - x^2}}$ becomes infinite for $x = a$. Therefore, by $(B)$,

$$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \underset{\epsilon=0}{\text{limit}} \int_0^{a-\epsilon} \frac{dx}{\sqrt{a^2 - x^2}} = \underset{\epsilon=0}{\text{limit}} \left[\text{arc sin}\frac{x}{a}\right]_0^{a-\epsilon}$$

$$= \underset{\epsilon=0}{\text{limit}} \left[\text{arc sin}\left(1 - \frac{\epsilon}{a}\right)\right] = \text{arc sin } 1 = \frac{\pi}{2}. \quad \textit{Ans.}$$

Ex. 2. Find $\int_0^1 \frac{dx}{x}$.

*Solution.* Here $\frac{1}{x}$ becomes infinite for $x = 0$. Therefore, by $(A)$,

$$\int_0^1 \frac{dx}{x} = \underset{\epsilon=0}{\text{limit}} \int_\epsilon^1 \frac{dx}{x} = \underset{\epsilon=0}{\text{limit}} \left(\frac{1}{\epsilon} - 1\right).$$

In this case there is no limit and therefore the integral does not exist.

If $c$ lies between $a$ and $b$ and $\phi(x)$ is continuous except at $x = c$, then, $\epsilon$ and $\epsilon'$ being positive numbers, the integral between $a$ and $b$ is *defined by*

$$(C)\qquad \int_a^b \phi(x)\,dx = \underset{\epsilon=0}{\text{limit}} \int_a^{c-\epsilon} \phi(x)\,dx + \underset{\epsilon'=0}{\text{limit}} \int_{c+\epsilon'}^b \phi(x)\,dx,$$

provided each separate limit is a definite quantity.

Ex. 1. Find $\int_0^{3a} \frac{2\,xdx}{(x^2 - a^2)^{\frac{2}{3}}}$.

*Solution.* Here the function to be integrated becomes infinite for $x = a$, i.e. for a value of $x$ between the limits of integration 0 and $3a$. Hence the above definition $(C)$ must be employed. Thus,

$$\int_0^{3a} \frac{2\,xdx}{(x^2 - a^2)^{\frac{2}{3}}} = \underset{\epsilon=0}{\text{limit}} \int_0^{a-\epsilon} \frac{2\,xdx}{(x^2 - a^2)^{\frac{2}{3}}} + \underset{\epsilon'=0}{\text{limit}} \int_{a+\epsilon'}^{3a} \frac{2\,xdx}{(x^2 - a^2)^{\frac{2}{3}}}$$

$$= \underset{\epsilon=0}{\text{limit}} \left[3\,(x^2 - a^2)^{\frac{1}{3}}\right]_0^{a-\epsilon} + \underset{\epsilon'=0}{\text{limit}} \left[3\,(x^2 - a^2)^{\frac{1}{3}}\right]_{a+\epsilon'}^{3a}$$

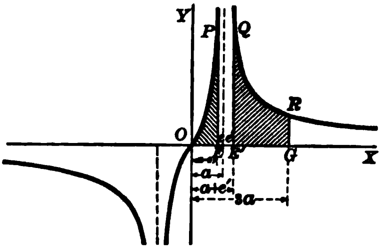
$$= \underset{\epsilon=0}{\text{limit}} \left[3\sqrt[3]{(a-\epsilon)^2 - a^2} + 3\,a^{\frac{2}{3}}\right] + \underset{\epsilon'=0}{\text{limit}} \left[3\sqrt[3]{8\,a^2} - 3\sqrt[3]{(a+\epsilon')^2 - a^2}\right]$$

$$= 3\,a^{\frac{2}{3}} + 6\,a^{\frac{2}{3}} = 9\,a^{\frac{2}{3}}. \quad \textit{Ans.}$$

To interpret this geometrically, let us plot the graph, i.e. the locus of

$$y = \frac{2x}{(x^2 - a^2)^{\frac{2}{3}}},$$

and note that $x = a$ is an asymptote.



area $OPE = \int_0^{a-\epsilon} \frac{2x\,dx}{(x^2 - a^2)^{\frac{2}{3}}}$

$$= 3\sqrt[3]{(a-\epsilon)^2 - a^2} + 3a^{\frac{2}{3}}.$$

Now as $PE$ moves to the right towards the asymptote, i.e. as $\epsilon$ approaches zero,

$$3\sqrt[3]{(a-\epsilon)^2 - a^2} + 3a^{\frac{2}{3}}$$

is always finite, and $\displaystyle\lim_{\epsilon = 0}\left[3\sqrt[3]{(a-\epsilon)^2 - a^2} + 3a^{\frac{2}{3}}\right] = 3a^{\frac{2}{3}}$,

which is also finite. As in Ex. 2, p. 361, $3a^{\frac{2}{3}}$ is called the area bounded by $OP$, the asymptote, and $OX$. Similarly,

$$\text{area } E'QRG = \int_{a+\epsilon'}^{3a} \frac{2x\,dx}{(x^2 - a^2)^{\frac{2}{3}}} = 3\sqrt[3]{8a^2} - 3\sqrt[3]{(a+\epsilon')^2 - a^2}$$

is always finite as $QE'$ moves to the left towards the asymptote, and as $\epsilon'$ approaches zero the result $6a^{\frac{2}{3}}$ is also finite. Hence $6a^{\frac{2}{3}}$ is called the area between $QR$, the asymptote, the ordinate $x = 3a$, and $OX$. Adding these results we get $9a^{\frac{2}{3}}$, which is then called the area to the right of $OY$ between the curve, the ordinate $x = 3a$, and $OX$.

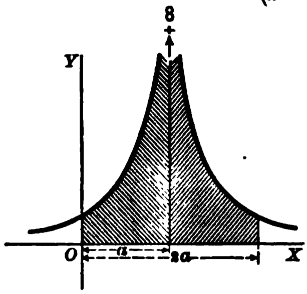
Ex. 2. Find $\displaystyle\int_0^{2a} \frac{dx}{(x-a)^2}$.

*Solution.* This function also becomes infinite between the limits of integration. Hence, by (*C*),

$$\int_0^{2a} \frac{dx}{(x-a)^2} = \lim_{\epsilon = 0} \int_0^{a-\epsilon} \frac{dx}{(x-a)^2} + \lim_{\epsilon' = 0} \int_{a+\epsilon'}^{2a} \frac{dx}{(x-a)^2}$$

$$= \lim_{\epsilon = 0}\left[-\frac{1}{x-a}\right]_0^{a-\epsilon} + \lim_{\epsilon' = 0}\left[-\frac{1}{x-a}\right]_{a+\epsilon'}^{2a}$$

$$= \lim_{\epsilon = 0}\left(\frac{1}{\epsilon} - \frac{1}{a}\right) + \lim_{\epsilon' = 0}\left(-\frac{1}{a} + \frac{1}{\epsilon'}\right).$$



In this case the limits do not exist and the integral has no meaning.

If we plot the graph of this function and note the limits, the condition of things appears very much the same as in the last example. It turns out, however, that the shaded portion cannot be properly spoken of as an area, and the integral sign has no meaning in this case.

That it is important to note whether or not the given function becomes infinite within the limits of integration will appear at once if we apply our integration formula without any investigation. Thus,

$$\int_0^{2a} \frac{dx}{(x-a)^2} = \left[-\frac{1}{x-a}\right]_0^{2a} = -\frac{2}{a},$$

a result which is absurd in view of the above discussions.

## EXAMPLES

1. $\int_2^3 6x^2dx = 38.$

2. $\int_0^a (a^2x - x^3)\,dx = \frac{a^4}{4}.$

3. $\int_1^4 \frac{dx}{x^{\frac{3}{2}}} = 1.$

4. $\int_1^e \frac{dx}{x} = 1.$

5. $\int_0^1 (x^2 - 2x + 2)(x - 1)\,dx = -\frac{3}{4}.$

6. $\int_0^1 \frac{dx}{\sqrt{3 - 2x}} = \sqrt{3} - 1.$

7. $\int_0^2 \frac{x^3dx}{x + 1} = \frac{8}{3} - \log 3.$

8. $\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{2 - 3x^2}} = \frac{\pi}{4\sqrt{3}}.$

9. $\int_2^3 \frac{3x\,dx}{2\sqrt[4]{x^2 - 4}} = \sqrt[4]{125}.$

10. $\int_0^1 \frac{dy}{y^2 - y + 1} = \frac{2\pi}{3\sqrt{3}}.$

11. $\int_2^3 \frac{t\,dt}{1 + t^2} = \frac{\log 2}{2}.$

12. $\int_1^\infty \frac{dx}{x^2(1 + x)} = 1 - \log 2.$

13. $\int_0^{+\infty} \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}.$

14. $\int_1^\infty \frac{dt}{t(1 - t)^2} = \log 2 - \frac{1}{2}.$

15. $\int_0^{\frac{\pi}{2}} \sin\phi\,d\phi = 1.$

16. $\int_0^{2\pi} \text{vers}^2\,\theta\,d\theta = 3\pi.$

17. $\int_0^{\frac{\pi}{4}} \sec^4\theta\,d\theta = \frac{4}{3}.$

18. $\int_0^1 \text{arc}\sin x\,dx = \frac{\pi}{2} - 1.$

19. $\int_1^e x\log x\,dx = \frac{e^2 + 1}{4}.$

20. $\int_0^{+\infty} e^{-x}dx = 1.$

21. $\int_0^1 \text{arc}\tan x\,dx = \frac{\pi}{4} - \sqrt{2}.$

22. $\int_0^{2r} \frac{\sqrt{2r}}{\sqrt{x}}\,dx = 4r.$

23. $\int_0^5 (\tfrac{3}{5}\sqrt{t} - \tfrac{3}{25}t^2)\,dt = 2\sqrt{5} - 5.$

24. $\int_0^r \frac{r\,dx}{\sqrt{r^2 - x^2}} = \frac{\pi r}{2}.$

25. $\int_0^{2r} \frac{2\sqrt{2r}\,dy}{\sqrt{2r - y}} = 8r.$

26. $\int_{-b}^b \frac{\pi}{a^4}(y^2 - b^2)^4\,dy = \frac{256\pi b^9}{315a^4}.$

27. $2a\int_0^\pi (2 + 2\cos\theta)^{\frac{1}{2}}d\theta = 8a.$

28. $\int_0^{\frac{\pi}{2}} \sin^3\alpha\cos^2\alpha\,d\alpha = \frac{2}{15}.$

29. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan\alpha\,d\alpha = 0.$

30. $\int_0^1 \log y\,dy = -1.$

31. $\int_0^1 x\log x\,dx = -\frac{1}{4}.$

32. $\int_0^1 x^2\log x\,dx = -\frac{1}{9}.$

33. $\int_0^{\frac{\pi}{2}} \alpha^2\sin\alpha\,d\alpha = \pi - 2.$

34. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec\theta\,d\theta = \log\left(\frac{1 + \sqrt{2}}{\sqrt{3}}\right).$

35. $\int_0^{\frac{\pi}{2}} \frac{d\alpha}{2 + \cos\alpha} = \frac{\pi}{3\sqrt{3}}.$

36. $\int_0^{\frac{\pi}{2}} \frac{\cos\theta\,d\theta}{1 + \sin^2\theta} = \frac{\pi}{4}.$

**218. Change in limits corresponding to change in the variable.** When integrating in Chapter XXVII by the substitution of a new variable it was often found quite troublesome to translate our result back into the original variable. When integrating between limits, however, we may avoid the restoration of the original variable by changing the limits to correspond with the new variable.* This process will now be illustrated by an example.

**Ex. 1.** Find $\int_0^a \sqrt{a^2 - x^2}\,dx$, assuming $x = a\sin\theta$.

*Solution.* $\sqrt{a^2 - x^2}\,dx = \sqrt{a^2 - a^2\sin^2 a}\cdot\cos\theta d\theta = a^2\cos^2\theta d\theta.$

When $x = a,\ a = a\sin\theta$, i.e. $\theta = \frac{\pi}{2}$;

and when $x = 0,\ 0 = a\sin\theta$, i.e. $\theta = 0$.

$$\therefore \int_0^a \sqrt{a^2 - x^2}\,dx = \int_0^{\frac{\pi}{2}} a^2\cos^2\theta d\theta = \left[\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}. \quad \textit{Ans.}$$

By this change in limits we mean that as $\theta$ increases from 0 to $\frac{\pi}{2}$, $x$ will increase from 0 to $a$.

EXAMPLES

| | |
|---|---|
| 1. $\int_0^4 \frac{dx}{1+\sqrt{x}} = 4 - 2\log 3.$ | Assume $\sqrt{x} = z$. |
| 2. $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}.$ | Assume $x = az$. |
| 3. $\int_{\frac{1}{3}}^1 \frac{(x - x^3)^{\frac{1}{3}}dx}{x^4} = 6.$ | Assume $x = \frac{1}{z}$. |
| 4. $\int_0^1 \frac{xdx}{\sqrt{1 - x^2}} = 1.$ | Assume $\sqrt{1 - x^2} = z$. |
| 5. $\int_0^{\frac{\pi}{2}} \sin a\cos^2 a da = \frac{1}{3}.$ | Assume $\sin a = z$. |
| 6. $\int_0^{\frac{\pi}{4}} \frac{(\sin\theta + \cos\theta)\,d\theta}{3 + \sin 2\theta} = \frac{\log 3}{4}.$ | Assume $\sin\theta - \cos\theta = z$. |
| 7. $\int_0^1 \frac{dx}{e^x + e^{-x}} = \arctan e - \frac{\pi}{4}.$ | Assume $e^x = z$. |
| 8. $\int_0^a \frac{dx}{\sqrt{ax - x^2}} = \pi.$ | Assume $x = a\sin^2 z$. |

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* The relation between the old and the new variable should be such that to each value of one within the limits of integration there is always one, and only one, finite value of the other. When one is given as a many-valued function of the other, care must be taken to choose the right values.

9. $\int_3^{29}\frac{(x-2)^{\frac{2}{3}}dx}{(x-2)^{\frac{2}{3}}+3} = 8 + \frac{3\sqrt{3}}{2}\pi.$ Assume $x - 2 = z^3$.

10. $\int_0^{\log 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx = 4 - \pi.$ Assume $e^x - 1 = z^2$.

11. $\int_1^4\frac{ydy}{\sqrt{2+4y}} = \frac{3\sqrt{2}}{2}.$ Assume $2 + 4y = z^2$.

12. $\int_0^{\pi}\frac{dt}{3+2\cos t} = \frac{\pi}{\sqrt{5}}.$ Assume $z = \tan\frac{t}{2}$.

13. $\int_0^{\log 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx = 4 - \pi.$ Assume $e^x - 1 = z^2$.

14. $\int_0^{a}\frac{x^3dx}{(a^2+x^2)^{\frac{3}{2}}} = \frac{3\sqrt{2}-4}{2}a.$ Assume $x = a\tan z$.

15. $\int_0^{a}y^2\sqrt{a^2-y^2}\,dy = \frac{\pi a^4}{16}.$ Assume $y = a\sin z$.

16. $\int_0^{1}\sqrt{2t+t^2}\,dt = \sqrt{3} - \frac{1}{2}\log(2+\sqrt{3}).$ Assume $t + 1 = z$.

17. $\int_0^{\log 2}\sqrt{e^x-1}\,dx = \frac{4-\pi}{2}.$ Assume $e^x + 1 = z$.

18. $\int_0^{\frac{\pi}{4}}\frac{\sin\theta+\cos\theta}{3+2\sin 2\theta}d\theta = \frac{1}{4}\log 3.$ Assume $\sin\theta - \cos\theta = z$.

19. $\int_1^{2+\sqrt{5}}\frac{(x^2+1)dx}{x\sqrt{x^4+7x^2+1}} = \log 3.$ Assume $x - \frac{1}{x} = z$.

## CHAPTER XXX

### INTEGRATION A PROCESS OF SUMMATION

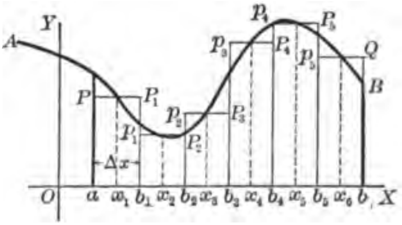
**219. Introduction.** Thus far we have defined integration as the *inverse of differentiation.* In a great many of the applications of the Integral Calculus, however, it is preferable to define integration as a *process of summation.* In fact the Integral Calculus was invented in the attempt to calculate the area bounded by curves by supposing the given area to be divided up into an "infinite number of infinitesimal parts called *elements*, the sum of all these elements being the area required." Historically the integral sign is merely the long *S*, used by early writers to indicate "sum."

This new definition, as amplified in the next section, is of fundamental importance, and it is essential that the student should thoroughly understand what is meant in order to be able to apply the Integral Calculus to practical problems.

**220. The definite integral defined as the limit of a sum of differential expressions.** Assume $\phi(x)dx$ as the differential of $f(x)$. Then by § 209, p. 356,

$$(A) \qquad \int_a^b \phi(x)\,dx = f(b) - f(a)$$

gives the area bounded by the curve $y = \phi(x)$ ($AB$ in figure), the axis of $X$, and the ordinates $x = a$ and $x = b$.* Now suppose the segment $ab$ to be divided into a number of equal parts, say 6, each equal to $\Delta x$, at points whose abscissas are $b_1$, $b_2$, $b_3$, $b_4$, $b_5$. Erect the ordinates at these points and apply the *Theorem of Mean Value* (44), p. 168, to each interval, noting that here $\phi(x)$ takes the place of $\phi'(x)$.



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* In the figure $a$, $b$, $b_1$, $x_1$, $x_2$, etc., denote the abscissas of the points under which they are written.

Applying (44) to the first interval ($a = a$, $b = b_1$, and $x_1$ lies between $a$ and $b_1$ as shown in figure), we have

$$\frac{f(b_1) - f(a)}{b_1 - a} = \phi(x_1),$$

or, since

$$b_1 - a = \Delta x,$$

$$f(b_1) - f(a) = \phi(x_1)\Delta x.$$

Applying (44) in the same way to each one of the remaining five segments, we get

$$\begin{aligned} f(b_1) - f(a) &= \phi(x_1)\Delta x, \\ f(b_2) - f(b_1) &= \phi(x_2)\Delta x, \\ f(b_3) - f(b_2) &= \phi(x_3)\Delta x, \\ f(b_4) - f(b_3) &= \phi(x_4)\Delta x, \\ f(b_5) - f(b_4) &= \phi(x_5)\Delta x, \\ f(b) - f(b_5) &= \phi(x_6)\Delta x, \end{aligned}$$

respectively.

Adding these six equations,

$$(B) \qquad f(b) - f(a) = \phi(x_1)\Delta x + \phi(x_2)\Delta x + \phi(x_3)\Delta x + \phi(x_4)\Delta x + \phi(x_5)\Delta x + \phi(x_6)\Delta x.$$

But

$$\phi(x_1)\Delta x = \text{area of first rectangle},$$

$$\phi(x_2)\Delta x = \text{area of second rectangle, etc.}$$

Hence the sum on the right-hand side of ($B$) equals the area of the figure bounded by the zigzag line $PP_1p_1P_2p_2P_3p_3P_4p_4P_5p_5Q$, the axis of $X$, and the ordinates $x = a$ and $x = b$. It is also evident that the area of this figure equals

$$f(b) - f(a),$$

no matter into how many equal parts the interval $[a, b]$ may be divided. Hence for *any* number $n$ of equal parts

$$(C) \qquad f(b) - f(a) = \phi(x_1)\Delta x + \phi(x_2)\Delta x + \cdots + \phi(x_n)\Delta x,$$

$$(D) \qquad \text{where } \Delta x = \frac{b-a}{n}.$$

When the number of segments ($= n$) into which the interval $[a, b]$ is divided increases without limit, equations ($C$) and ($D$) still hold true, but $\Delta x$ becomes $dx$, i.e. a variable whose limit is zero (§ 30, p. 21).

$$\therefore\quad f(b)-f(a)=\lim_{n=\infty}[\phi(x_1)\,dx+\phi(x_2)\,dx+\cdots+\phi(x_n)\,dx],$$

or, by $(A)$,

$$(E)\qquad \int_a^b \phi(x)\,dx=\lim_{n=\infty}[\phi(x_1)\,dx+\phi(x_2)\,dx+\cdots+\phi(x_n)\,dx].$$

This exhibits our definite integral as the *limit of a sum of differential expressions.* Each one of the differential expressions $\phi(x_1)\,dx$, $\cdots$, $\phi(x_n)\,dx$ is called an *element* of the integral.

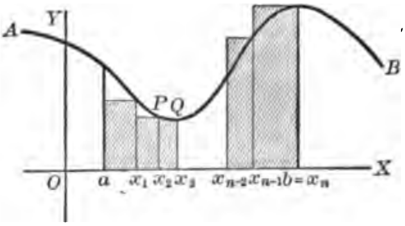
The above result also illustrates very clearly the definition of the definite integral as the area under the curve. For as $n$ increases without limit the sum

$$\phi(x_1)\,dx+\phi(x_2)\,dx+\cdots+\phi(x_n)\,dx$$

always represents the area under the zigzag line, and evidently the figure bounded by the zigzag line, the end ordinates, and $OX$ approaches the area under the curve as a limit.

We may, however, attack the problem in a much more general way, for it is geometrically evident that a subdivision of the given area may be made in an infinite variety of ways such that the continuation of the process will lead in the limit to the area desired. For example, let us choose within the interval $[a, b]$, $n-1$ abscissas, $x_1$, $x_2$, $\cdots$, $x_{n-1}$, *in any manner whatever,* and erect ordinates at the corresponding points to the curve. Then the area is divided into $n$ portions such as $x_1P$, $x_2Q$, etc. Denote the lengths of the subdivisions on $OX$ as follows:

$$x_1-a=\Delta x_1,$$
$$x_2-x_1=\Delta x_2,$$
$$\cdot\quad\cdot\quad\cdot$$
$$x_{n-1}-x_{n-2}=\Delta x_{n-1},$$
$$b-x_{n-1}=\Delta x_n.$$



Then evidently the area of the segment $x_3Q$, for example, equals approximately that of the rectangle whose base is $\Delta x_3$ and altitude $Qx_3$, or $\phi(x_3)$. Carrying out this idea for each portion, we have as a result that

$$\phi(x_1)\Delta x_1+\phi(x_2)\Delta x_2+\cdots+\phi(x_{n-1})\Delta x_{n-1}+\phi(x_n)\Delta x_n)$$

gives an area approximately equal to the area required. And now it is geometrically obvious that the *limit* of the above sum is the area under the curve if the process of subdivision of $[a, b]$ be continued according to some law by which each division on $x$ approaches the limit zero. That is,

$$\lim_{n=\infty}\sum_{i=1}^{n}\phi(x_i)\,\Delta x_i{}^{*} = \int_a^b \phi(x)\,dx.$$

This general discussion shows that *integration between limits is a process of summation*, in which, however, the definite integral appears not as a simple sum but as the *limit of a sum, the number of terms increasing without limit, and each term separately approaching the limit zero.*

In order to replace the intuitional point of view that we have so far adopted in the text by a rigorous and general analytical proof, proceed as follows.

Divide the interval $[a, b]$ into any number of parts $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$, and let $x'_i$ be *any* abscissa in the segment $\Delta x_i$, the extremities of this segment being denoted by $x_i$ and $x_{i+1}$. Also suppose $\overline{x_i}$ to be a value in the interval $[x_i, x_{i+1}]$ determined by the Theorem of Mean Value (as in the first part of this section), so that

$$f(x_{i+1}) - f(x_i) = \phi(\overline{x_i})\,\Delta x_i.$$

Then, *exactly as before*, the sum

$$(F) \qquad \sum_{1}^{n}\phi(\overline{x_i})\,\Delta x_i$$

equals the required area. And while the corresponding sum

$$(G) \qquad \sum_{1}^{n}\phi(x'_i)\Delta x_i$$

does not also give the area, nevertheless we may show that the two sums $(F)$ and $(G)$ approach equality when $n$ increases without limit. For the difference $\phi(\overline{x_i}) - \phi(x'_i)$ does not exceed in numerical value the difference of the greatest and smallest ordinates in $\Delta x_i$. And furthermore, it is always possible[†] to make all these differences less in numerical value than any assignable positive number $\epsilon$, however small, by continuing the process of subdivision far enough, i.e. by choosing $n$ sufficiently large. Hence for such a choice of $n$ the difference of the sums $(F)$ and $(G)$ is less in numerical value than $\epsilon(b-a)$, i.e. less than any assignable positive quantity, however small. Accordingly as $n$ increases without limit, the sums $(F)$ and $(G)$ approach equality, and since $(F)$ is always equal to the area, the fundamental result follows that

$$\int_a^b \phi(x)\,dx = \lim_{n=\infty}\sum_{1}^{n}\phi(x'_i)\,\Delta x_i,$$

in which the interval $[a, b]$ is subdivided in any manner whatever, and $x'_i$ is any abscissa in the corresponding subdivision.

---

[*] Any term of this sum is sometimes called an *element of the area*, for each one represents the area of one of the rectangles forming the whole figure.

[†] That such is the case is shown in advanced works on the Calculus.

The process of evaluation of the limit of this sum is accordingly often spoken of as *summing up an infinite number of infinitely small quantities*, but the phrase has no meaning except in the above sense.

We may apply to great advantage this fruitful idea of summing up an infinite number of infinitely small quantities to a large number of the problems of the Integral Calculus. In order to obtain solutions by this method the following steps are in general to be taken.

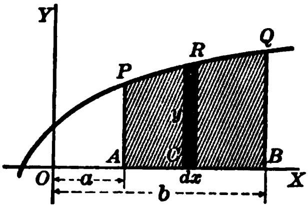
*First step.* Find a differential expression for any one of the infinitesimal quantities (elements) composing the quantity to be calculated,* and reduce it to a form involving only a single variable.

*Second step.* Integrate this differential expression (i.e. sum up all the elements) between the limits given by the conditions in the problem.

**221. Areas of plane curves. Rectangular coördinates.** It was shown in § 209, p. 356, that the area between a curve, the axis of $X$, and the ordinates $x = a$ and $x = b$ is given by the formula

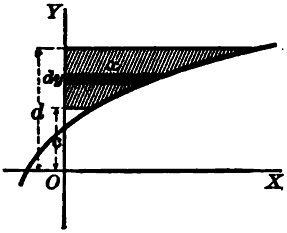
$$(A) \qquad \textbf{area} = \int_a^b y\,dx,$$

the value of $y$ in terms of $x$ being substituted from the equation of the curve.



Considered as a process of summation, it is customary to look upon this operation as follows (see p. 367).

Consider any strip (as $CR$) as an element of the area. Regarding it as a rectangle of altitude $y$ and infinitesimal base $dx$, its area is equal to $ydx$, and summing up all such strips between the ordinates $AP$ and $BQ$ gives the area $ABQP$.



Similarly it may be shown that the area between a curve, the axis of $Y$, and the lines $y = c$ and $y = d$ is given by the formula

$$(B) \qquad \textbf{area} = \int_c^d x\,dy,$$

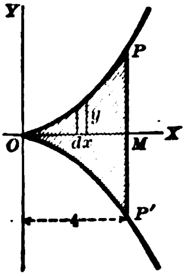
the value of $x$ in terms of $y$ being substituted from the equation of the curve.

---

* If an area is wanted, find a differential expression for an element of the area; if a length, find it for an element of the length; if a volume, find it for an element of the volume, etc.

Ex. 1. Find the area included between the semicubical parabola $y^2 = x^3$ and the line $x = 4$.

*Solution.* Let us first find the area $OMP$, half of the required area $OPP'$. For the upper branch of the curve $y = \sqrt{x^3}$, and summing up all the strips between the limits $x = 0$ and $x = 4$, we get, by substituting in ($A$),



$$\text{area } OMP = \int_0^4 y\,dx = \int_0^4 x^{\frac{3}{2}}\,dx = \tfrac{64}{5} = 12\tfrac{4}{5}.$$

Hence area $OPP' = 2 \cdot \frac{64}{5} = 25\frac{3}{5}$.

If the unit of length is one inch, the area of $OPP$ is $25\frac{3}{5}$ square inches.

Note. For the lower branch $y = -x^{\frac{3}{2}}$; hence

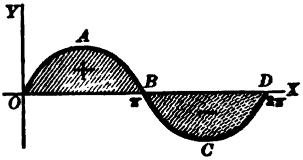
$$\text{area } OMP' = \int_0^4 (-x^{\frac{3}{2}})\,dx = -12\tfrac{4}{5}.$$

*This area lies below the axis of x and has a negative sign* because the ordinates are negative.

In finding the area $OMP$ above, *the result was positive* because the ordinates were positive, *the area lying above the axis of x.*

The above result, $25\frac{3}{5}$, was the total area regardless of sign. As we shall *illustrate* in the next example, it is important to note the sign of the area when the curve crosses the axis of $X$ within the limits of integration.

Ex. 2. Find the area of one arch of the sine curve $y = \sin x$.



*Solution.* Placing $y = 0$ and solving for $x$, we find

$$x = 0,\ \pi,\ 2\pi, \text{ etc.}$$

Substituting in ($A$), p. 371,

$$\text{area } OAB = \int_a^b y\,dx = \int_0^{\pi} \sin x\,dx = 2.$$

Also,
$$\text{area } BCD = \int_a^b y\,dx = \int_{\pi}^{2\pi} \sin x\,dx = -2,$$

and
$$\text{area } OABCD = \int_a^b y\,dx = \int_0^{2\pi} \sin x\,dx = 0.$$

This last result takes into account the signs of the two separate areas composing the whole. The total area regardless of these signs equals 4.

Ex. 3. Find the area included between the parabola $x^2 = 4\,ay$ and the witch

$$y = \frac{8\,a^3}{x^2 + 4\,a^2}.$$

*Solution.* To determine the limits of integration, we solve the equations simultaneously to find where the curves intersect. The coördinates of $A$ are found to be $(-2\,a, a)$, and of $B$ $(2\,a, a)$.

It is seen from the figure that

$$\text{area } AOCB = \text{area } DECBA - \text{area } DECOA.$$

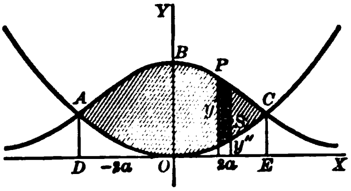
But $\qquad \text{area } DECBA = 2 \times \text{area } OECB = 2\int_0^{2a} \frac{8a^3dx}{x^2+4a^2} = 2\pi a^2,$

and $\qquad \text{area } DECOA = 2 \times \text{area } OEC = 2\int_0^{2a} \frac{x^2}{4a}dx = \frac{4a^2}{3}.$

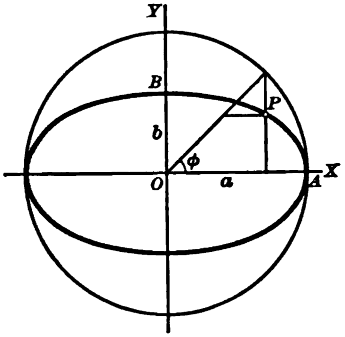
Hence $\qquad \text{area } AOCB = 2\pi a^2 - \frac{4a^2}{3} = 2a^2(\pi - \frac{2}{3}).$ *Ans.*

Another method is to consider the strip $PS$ as an element of the area. If $y'$ is the ordinate corresponding to the witch, and $y''$ to the parabola, the differential expression for the area of the strip $PS$ equals $(y' - y'')dx$. Substituting the values of $y'$ and $y''$ in terms of $x$ from the given equations, we get

$$\begin{aligned} \text{area } AOCB &= 2 \times \text{area } OCB \\ &= 2\int_0^{2a}(y' - y'')\,dx \\ &= 2\int_0^{2a}\left(\frac{8a^3}{x^2+4a^2} - \frac{x^2}{4a}\right)dx \\ &= 2a^2(\pi - \tfrac{2}{3}). \end{aligned}$$



Ex. 4. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$



*Solution.* To find the area of the quadrant $OAB$, the limits are $x = 0$, $x = a$; and

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Hence, substituting in $(A)$, p. 371,

$$\begin{aligned} \text{area } OAB &= \frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}}dx \\ &= \left[\frac{bx}{2a}(a^2 - x^2)^{\frac{1}{2}} + \frac{ab}{2}\arcsin\frac{x}{a}\right]_0^a \\ &= \frac{\pi ab}{4}. \qquad [B], \text{ p. } 345. \end{aligned}$$

Therefore the entire area of the ellipse equals $\pi ab$.

**222. Area when curve is given in parametric form.** Let the equation of the curve be given in the parametric form

$$x = f(t), \quad y = \phi(t).$$

We then have $\qquad dy = \phi(t)$ and $dx = f'(t)\,dt$,

which substituted[*] in $(A)$, p. 371, gives

$$(A) \qquad \text{area} = \int_{t_1}^{t_2} \phi(t) f'(t)\,dt,$$

where $\qquad t = t_1$ when $x = a$, and $t = t_2$ when $x = b$.

We may employ this formula $(A)$ when finding the area under a curve given in parametric form. Or we may find $y$ and $dx$ from the parametric equations of the curve in terms of $t$ and $dt$ and then substitute the results directly in $(A)$, p. 371.

Thus, in finding the area of the ellipse in Ex. 4, p. 373, it would have been simpler to use the parametric equations of the ellipse

$$x = a \cos \phi, \quad y = b \sin \phi,$$

where the eccentric angle $\phi$ is the parameter (§ 79, p. 94).

Here $\qquad y = b \sin \phi$ and $dx = - a \sin \phi d\phi$.

When $\qquad x = 0$, $\phi = \frac{\pi}{2}$; and when $x = a$, $\phi = 0$.

Substituting these in $(A)$, p. 371, we get

$$\text{area } OAB = \int_0^a y dx = - \int_{\frac{\pi}{2}}^{0} ab \sin^2 \phi d\phi = \frac{\pi ab}{4}.$$

Hence the entire area equals $\pi ab$. *Ans.*

**EXAMPLES**

1. Find the area bounded by the line $y = 5x$, the axis of $X$, and the ordinate $x = 2$. *Ans.* 10.

2. Find the area bounded by the parabola $y^2 = 4x$, the axis of $X$, and the lines $x = 4$ and $x = 9$. *Ans.* $25\frac{1}{3}$.

3. Find the area bounded by the parabola $y^2 = 4x$, the axis of $Y$, and the lines $y = 4$ and $y = 6$. *Ans.* $12\frac{2}{3}$.

4. Find the area of the circle $x^2 + y^2 = r^2$. *Ans.* $\pi r^2$.

5. Find the area between the equilateral hyperbola $xy = a^2$, the axis of $X$, and the ordinates $x = a$ and $x = 2a$. *Ans.* $a^2 \log 2$.

6. Find the area between the curve $y = 4 - x^2$ and the axis of $X$. *Ans.* $10\frac{2}{3}$.

7. Find the area intercepted between the coördinate axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $\frac{a^2}{6}$.

---

[*] For a rigorous proof of this substitution the student is referred to more advanced treatises on the Calculus.

**8.** Find the area bounded by the semicubical parabola $y^2 = x^3$, the axis of $Y$, and the line $y = 4$. *Ans.* $\frac{3}{5}\sqrt[3]{1024}$.

**9.** Find the area between the catenary $y = \frac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right]$, the axis of $Y$, and the line $x = a$.
*Ans.* $\frac{a^2}{2e}[e^2 - 1]$.

**10.** Find the area between the curve $y = \log x$, the axis of $X$, and the ordinates $x = 1$ and $x = a$. *Ans.* $a(\log a - 1) + 1$.

**11.** Find the entire area of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

*Ans.* $\frac{3\pi ab}{4}$.

**12.** Find the entire area of the curve $a^2y^2 = x^3(2a - x)$. *Ans.* $\pi a^2$.

**13.** Find the area bounded by the curves

$$x(y - e^x) = \sin x \text{ and } 2xy = 2\sin x + x^3,$$

the axis of $Y$, and the ordinate $x = 1$. *Ans.* $\int_0^1 (e^x - \frac{1}{2}x^2)\,dx = e - \frac{7}{6} = 1.55 + \cdots$.

**14.** Find the area between the witch $y = \frac{8a^3}{x^2 + 4a^2}$ and the axis of $X$, its asymptote. *Ans.* $4\pi a^2$.

**15.** Find the area between the cissoid $y^2 = \frac{x^3}{2a - x}$ and its asymptote, the line $x = 2a$. *Ans.* $3\pi a^2$.

**16.** Prove that the area bounded by a parabola and one of its double ordinates equals two thirds of the circumscribing rectangle having the double ordinate as one side.

**17.** Find the area included between the two parabolas $y^2 = 2px$ and $x^2 = 2py$.
*Ans.* $\frac{4p^2}{3}$.

**18.** Find the area included between the parabola $y^2 = 2x$ and the circle $y^2 = 4x - x^2$. *Ans.* 0.475.

**19.** Find the total area included between the curve $y = x^3$ and the line $y = 2x$.
*Ans.* 2.

**20.** Find an expression for the area bounded by the equilateral hyperbola $x^2 - y^2 = a^2$, the axis of $X$, and the diameter through any point $(x, y)$.
*Ans.* $\frac{a^2}{2}\log\frac{x + y}{a}$.

**21.** Find by integration the area of the triangle bounded by the axis of $Y$ and the lines $2x + y + 8 = 0$ and $y = -4$. *Ans.* 4.

**22.** Find the area of the circle

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta; \end{cases}$$

$\theta$ being the parameter. *Ans.* $\pi r^2$.

**23.** Find the area of one arch of the cycloid

$$\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta); \end{cases}$$

$\theta$ being the parameter.

*Hint.* Since $x$ varies from 0 to $2\pi a$, $\theta$ varies from 0 to $2\pi$.

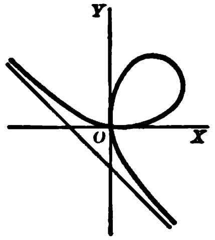
*Ans.* $3\pi a^2$; that is, three times the area of the generating circle.

**24.** Find the area of the hypocycloid

$$\begin{cases} x = a\cos^3\theta, \\ y = a\sin^3\theta; \end{cases}$$

$\theta$ being the parameter.

*Ans.* $\frac{3\pi a^2}{8}$; that is, three eighths of the area of the circumscribing circle.



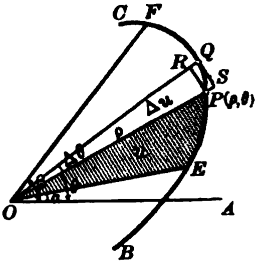
**25.** Find the area of the loop of the folium of Descartes $x^3 + y^3 = 3axy$.

*Hint.* Let $y = tx$; then $x = \frac{3at}{1+t^3}$,

$$y = \frac{3at^2}{1+t^3}, \text{ and } dx = \frac{1-2t^3}{(1+t^3)^2}3a\,dt.$$

The limits for $t$ are 0 and $\infty$.

**223. Areas of plane curves. Polar coördinates.** Let $BC$ be a curve whose equation is given in polar coördinates. Let $(\rho, \theta)$ be the coördinates of $P$, and assume $u$ as the measure of the area $OEP$.* When $\theta$ takes on a small increment $\Delta\theta$, $u$ takes on the increment $\Delta u$ (= area $OPQ$). Completing the circular sectors $OPR$ and $OSQ$, it is seen that



$$\text{area } OPR < \text{area } OPQ < \text{area } OSQ,$$

or, $$\tfrac{1}{2}\overline{OP}^2\cdot\Delta\theta\dagger < \Delta u < \tfrac{1}{2}\overline{OQ}^2\cdot\Delta\theta.$$

Dividing through by $\Delta\theta$,

$$\tfrac{1}{2}\overline{OP}^2 < \frac{\Delta u}{\Delta\theta} < \tfrac{1}{2}\overline{OQ}^2.\ddagger$$

Now let $\Delta\theta$ approach zero as a limit; then $OQ$ will approach $OP$ as a limit, and we get

$$\frac{du}{d\theta} = \frac{1}{2}\rho^2\left(=\frac{1}{2}\overline{OP}^2\right).$$

---

* Since we may suppose this area to be generated by a variable radius vector starting out from $OE$ and moving up to the position $OP$, $u$ will be a function of $\theta$ which vanishes when $\theta = \alpha$.

† The area of a circular sector $= \frac{1}{2}$ radius × arc $= \frac{1}{2} OP \times OP\Delta\theta$.

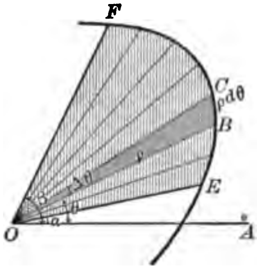
‡ In this figure $OP$ is less than $OQ$; if $OP$ is greater than $OQ$, simply reverse the inequality signs.

Or, using differentials, $du = \frac{1}{2}\rho^2 d\theta$,

this being the *differential of the area in polar coördinates.*

Integrating, we have $u = \frac{1}{2}\int \rho^2 d\theta.$

If we now apply the same line of reasoning as that followed in § 209, p. 356, the area of the sector $OEF$ may be calculated by means of the formula



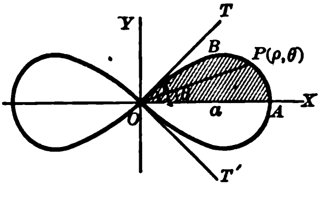
**(A)** $$\text{area} = \frac{1}{2}\int_\alpha^\beta \rho^2 d\theta,$$

the value of $\rho$ in terms of $\theta$ being substituted from the equation of the curve.

To look at this process as a summation, consider any sector $OBC$ as the element of area. Regarding it as a circular sector of radius $\rho$ and infinitesimal arc $\rho d\theta$, its area equals $\frac{1}{2}\rho^2 d\theta$. Summing up all such sectors between $OE$ and $OF$ gives the area $OEF$.

*Formula (A) may then be used for finding the area bounded by a polar curve and the radii vectors corresponding to $\theta = \alpha$ and $\theta = \beta$.*

Ex. 1. Find the entire area of the lemniscate $\rho^2 = a^2 \cos 2\theta$.



*Solution.* Since the figure is symmetrical with respect to both $OX$ and $OY$, the whole area = 4 times the area of $OAB$.

Since $\rho = 0$ when $\theta = \frac{\pi}{4}$, we see that if $\theta$ varies from 0 to $\frac{\pi}{4}$, the radius vector $OP$ sweeps over the area $OAB$. Hence, substituting in (A),

$$\text{entire area} = 4 \times \text{area } OAB = 4 \cdot \frac{1}{2}\int_\alpha^\beta \rho^2 d\theta = 2a^2\int_0^{\frac{\pi}{4}} \cos 2\theta d\theta = a^2;$$

that is, the area of both loops equals the area of a square constructed on $OA$ as one side.

### EXAMPLES

1. Find the area swept over in one revolution by the radius vector of the spiral of Archimedes, $\rho = a\theta$, starting with $\theta = 0$. *Ans.* $\frac{4\pi^3a^2}{3}$.

2. Find the area of one loop of the curve $\rho = a \cos 2\theta$. *Ans.* $\frac{\pi a^2}{8}$.

**3.** Show that the entire area of the curve $\rho = a \sin 2\theta$ equals one half the area of the circumscribed circle.

**4.** Find the entire area of the cardioid $\rho = a(1 - \cos\theta)$.

*Ans.* $\frac{3\pi a^2}{2}$; that is, six times the area of the generating circle.

**5.** Find the area of the circle $\rho = a\cos\theta$. *Ans.* $\frac{\pi a^2}{4}$.

**6.** Prove that the area of the three loops of $\rho = a \sin 3\theta$ equals one fourth of the area of the circumscribed circle.

**7.** Prove that the area generated by the radius vector of the spiral $\rho = e^{\theta}$ equals one fourth of the area of the square described on the radius vector.

**8.** Find the area of that part of the parabola $\rho = a\sec^2\frac{\theta}{2}$ which is intercepted between the curve and the latus rectum. *Ans.* $\frac{8a^2}{3}$.

**9.** Show that the area bounded by any two radii vectors of the hyperbolic spiral $\rho\theta = a$ is proportional to the difference between the lengths of these radii.

**10.** Find the area of the ellipse $\rho^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$. *Ans.* $\pi ab$.

**11.** Find the entire area of the curve $\rho = a(\sin 2\theta + \cos 2\theta)$. *Ans.* $\pi a^2$.

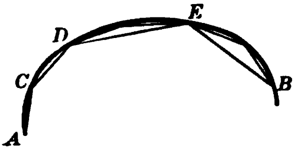
**12.** Find the area of one loop of the curve $\rho^2\cos\theta = a^2\sin 3\theta$.

*Ans.* $\frac{3a^2}{4} - \frac{a^2}{2}\log 2$.

**13.** Find the area below $OX$ within the curve $\rho = a\sin^3\frac{\theta}{3}$.

*Ans.* $(10\pi + 27\sqrt{3})\frac{a^2}{64}$.

**224. Length of a curve.** By the *length of a straight line* we commonly mean the number of times we can superpose upon it another straight line employed as a unit of length, as when the carpenter measures the length of a board by making end-to-end applications of his foot rule.



Since it is impossible to make a straight line coincide with an arc of a curve, we cannot measure curves in the same manner as we measure straight lines. We proceed then as follows.

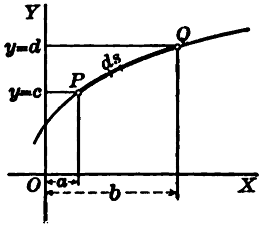
Divide the curve (as $AB$) into any number of parts in any manner whatever (as at $C$, $D$, $E$) and connect the adjacent points of division, forming chords (as $AC$, $CD$, $DE$, $EB$).

*The length of the curve is defined as the limit of the sum of the chords as the number of points of division increases without limit in such a way that at the same time each chord separately approaches zero as a limit.*

Since this limit will also be the measure of the length of some straight line, the finding of the length of a curve is also called "the rectification of the curve."

The student has already made use of this definition for the length of a curve in his Geometry. Thus the circumference of a circle is defined as the limit of the perimeter of the inscribed (or circumscribed) regular polygon when the number of sides increases without limit.

The method of the next section for finding the length of a plane curve is based on the above definition, and the student should note very carefully how it is applied.



**225. Lengths of plane curves. Rectangular coördinates.** We shall now proceed to express in analytical form the definition of the last section. Given the curve

$$y=f(x)$$

and the points $P(a, c)$, $Q(b, c)$ on it; to find the length of the arc $PQ$.

Take any number $(=n)$ of points on the curve between $P$ and $Q$, say $P', P'', \cdots, P^{(n)}$, and draw the chords $PP', P'P'', \cdots, P^{(n)}Q$. Consider any one of these chords, $P'P''$, for example, and let the coördinates of $P'$ and $P''$ be

$$P'(x', y') \text{ and } P'(x'+\Delta x', y'+\Delta y').$$

Then, as in § 102, p. 141,

$$P'P''=\sqrt{(\Delta x')^2+(\Delta y')^2},$$

or,

$$P'P''=\left[1+\left(\frac{\Delta y'}{\Delta x'}\right)^2\right]^{\frac{1}{2}}\Delta x'.$$

[Dividing inside the radical by $(\Delta x')^2$ and multiplying outside by $\Delta x'$.]

But from the Theorem of Mean Value, (42), p. 167 (if $\Delta y'$ is denoted by $f(b)-f(a)$ and $\Delta x'$ by $b-a$), we get

$$\frac{\Delta y'}{\Delta x'}=f'(x_1). \qquad x'<x_1<x'+\Delta x'$$

Substituting, we get

$$P'P''=[1+f'(x_1)^2]^{\frac{1}{2}}\Delta x'.$$

In the same manner we find

$$PP'=[1+f'(x_0)^2]^{\frac{1}{2}}\Delta x^{(0)},$$
$$P'P''=[1+f'(x_1)^2]^{\frac{1}{2}}\Delta x',$$
$$P''P'''=[1+f'(x_2)^2]^{\frac{1}{2}}\Delta x'',$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$P^{(n)}Q=[1+f'(x_n)^2]^{\frac{1}{2}}\Delta x^{(n)}.$$

The length of the inscribed broken line joining $P$ and $Q$ (sum of the chords) is then the sum of these expressions, and the length of the arc $PQ$ is therefore, by definition, the *limit* of the sum

$$[1+f'(x_0)^2]^{\frac{1}{2}}\Delta x^{(0)}+[1+f'(x_1)^2]^{\frac{1}{2}}\Delta x'+\cdots+[1+f'(x_n)^2]\,\Delta x^{(n)}$$

*as n increases without limit.* Hence, if we denote the length of the arc $PQ$ by $s$, we have, by § 220, p. 367, the **formula for the length of the arc,**

$$s=\int_a^b[1+f'(x)^2]^{\frac{1}{2}}dx, \text{ or,}$$

$$(A) \qquad s=\int_a^b\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx,$$

where $\frac{dy}{dx}$ must be found in terms of $x$ from the equation of the given curve.

Sometimes it is more convenient to use $y$ as the independent variable. To derive a formula to cover this case, we know from (33), p. 152, that

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}; \text{ hence } dx=\frac{dx}{dy}dy.$$

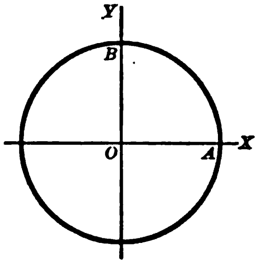
Substituting this value of $dx$ in ($A$), and noting that the corresponding $y$ limits are $c$ and $d$, we get* the **formula for the length of the arc,**

$$(B) \qquad s=\int_c^d\left[\left(\frac{dx}{dy}\right)^2+1\right]^{\frac{1}{2}}dy,$$

where $\frac{dx}{dy}$ in terms of $y$ must be found from the equation of the given curve.

---

* $s=\int_c^d\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}\frac{dx}{dy}dy=\int_c^d\left[\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dx}\right)^2\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}dy=\int_c^d\left[\left(\frac{dx}{dy}\right)^2+1\right]^{\frac{1}{2}}dy.$



**Ex. 1.** Find the length of the circle $x^2 + y^2 = r^2$.

***Solution.*** Differentiating, $\frac{dy}{dx} = -\frac{x}{y}$.

**Substituting** in $(A)$,

$$\text{arc } BA = \int_0^r \left[1 + \frac{x^2}{y^2}\right]^{\frac{1}{2}} dy$$

$$= \int_0^r \left[\frac{y^2 + x^2}{y^2}\right]^{\frac{1}{2}} dx = \int_0^r \left[\frac{r^2}{a^2 - x^2}\right]^{\frac{1}{2}} dx.$$

$\left[\begin{array}{l}\textbf{Substituting } y^2 = a^2 - x^2 \textbf{ from the equation of the}\\ \textbf{circle in order to get everything in terms of } x.\end{array}\right]$

$$\therefore \quad \text{arc } BA = r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = \left[r \text{ arc sin } \frac{x}{r}\right]_0^r = \frac{\pi r}{2}.$$

**Hence the total length equals $2\pi r$.** *Ans.*

## EXAMPLES

1. Find the length of the arc of the semicubical parabola $ay^2 = x^3$ from the origin to the ordinate $x = 5a$. *Ans.* $\frac{335a}{27}$.

2. Find the entire length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $6a$.

3. Rectify the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from $x = 0$ to the point $(x, y)$.

*Ans.* $\frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.

4. Find the length of one complete arch of the cycloid

$$x = r \text{ arc vers } \frac{y}{r} - \sqrt{2ry - y^2}. \qquad \textit{Ans. } 6a.$$

*Hint.* Use $(B)$. Here $\frac{dx}{dy} = \frac{y}{\sqrt{2ry - y^2}}$.

5. Find the length of the arc of the parabola $y^2 = 2px$ from the vertex to one extremity of the latus rectum. *Ans.* $\frac{p\sqrt{2}}{2} + \frac{p}{2}\log(1 + \sqrt{2})$.

6. Rectify the curve $9ay^2 = x(x - 3a)^2$ from $x = 0$ to $x = 3a$. *Ans.* $2a\sqrt{3}$.

7. Find the length in one quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

*Ans.* $\frac{a^2 + ab + b^2}{a + b}$.

8. Find the length between $x = a$ and $x = b$ of the curve $e^y = \frac{e^x + 1}{e^x - 1}$.

*Ans.* $\log \frac{e^{2b} - 1}{e^{2a} - 1} + a - b$.

9. The equations of the involute of a circle are

$$\begin{cases} x = a(\cos\theta + \theta \sin\theta), \\ y = a(\sin\theta - \theta\cos\theta). \end{cases}$$

Find the length of the arc from $\theta = 0$ to $\theta = \theta_1$. *Ans.* $\frac{1}{2}a\theta_1^2$.

**226. Lengths of plane curves. Polar coördinates.** Formulas (*A*) and (*B*) of the last section for finding the lengths of curves whose equations are given in rectangular coördinates involved the differential expressions

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx \text{ and } \left[\left(\frac{dx}{dy}\right)^2+1\right]^{\frac{1}{2}}dy.$$

In each case, if we introduce the differential of the independent variable inside the radical, they reduce to the form

$$[dx^2+dy^2]^{\frac{1}{2}}.$$

Let us now transform this expression into polar coördinates by means of the substitutions

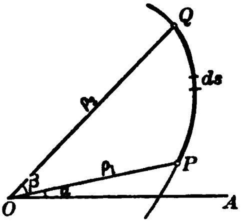
$$x=\rho\cos\theta,\quad y=\rho\sin\theta.$$

Then $\qquad dx=-\rho\sin\theta d\theta+\cos\theta d\rho,$

and $\qquad dy=\rho\cos\theta d\theta+\sin\theta d\rho,$

and we have

$$\begin{aligned}[dx^2+dy^2]^{\frac{1}{2}}&=[(-\rho\sin\theta d\theta+\cos\theta d\rho)^2+(\rho\cos\theta d\theta+\sin\theta d\rho)^2]^{\frac{1}{2}}\\&=[\rho^2d\theta^2+d\rho^2]^{\frac{1}{2}}.\end{aligned}$$



If the equation of the curve is

$$\rho=f(\theta),$$

then $\qquad d\rho=f'(\theta)\,d\theta=\frac{d\rho}{d\theta}d\theta.$

Substituting this in the above differential expression, we get

$$\left[\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}}d\theta.$$

If then $\alpha$ and $\beta$ are the limits of the independent variable $\theta$ corresponding to the limits in (*A*) and (*B*), p. 308, we get the **formula for the length of the arc,**

$$(A)\qquad s=\int_\alpha^\beta\left[\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}}d\theta,$$

where $\rho$ and $\frac{d\rho}{d\theta}$ in terms of $\theta$ must be substituted from the equation of the given curve.

In case it is more convenient to use $\rho$ as the independent variable, and the equation is in the form

$$\theta = \phi(\rho),$$

then

$$d\theta = \phi'(\rho)\,d\rho = \frac{d\theta}{d\rho}\,d\rho.$$

Substituting this in $[\rho^2 d\theta^2 + d\rho^2]^{\frac{1}{2}}$

gives

$$\left[\rho^2\left(\frac{d\theta}{d\rho}\right)^2 + 1\right]^{\frac{1}{2}} d\rho.$$

Hence, if $\rho_1$ and $\rho_2$ are the corresponding limits of the independent variable $\rho$, we get the **formula for the length of the arc,**

$$(B) \qquad s = \int_{\rho_1}^{\rho_2} \left[\rho^2\left(\frac{d\theta}{d\rho}\right)^2 + 1\right]^{\frac{1}{2}} d\rho,$$

where $\frac{d\theta}{d\rho}$ in terms of $\rho$ must be substituted from the equation of the given curve.

**Ex. 1.** Find the perimeter of the cardioid $\rho = a(1 + \cos\theta)$.

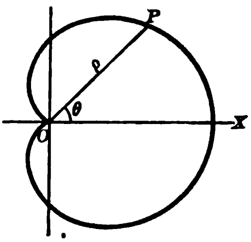
*Solution.* Here $\frac{d\rho}{d\theta} = -a\sin\theta$.

If we let $\theta$ vary from $\theta$ to $\pi$, the point $P$ will generate one half of the curve. Substituting in ($A$), p. 382,

$$\frac{s}{2} = \int_0^{\pi} [a^2(1 + \cos\theta)^2 + a^2\sin^2\theta]^{\frac{1}{2}}\,d\theta$$

$$= a\int_0^{\pi} (2 + 2\cos\theta)^{\frac{1}{2}}\,d\theta = 2a\int_0^{\pi} \cos\frac{\theta}{2}\,d\theta = 4a.$$

$\therefore \quad s = 8a.$ *Ans.*



## EXAMPLES

1. Find the length of the spiral of Archimedes, $\rho = a\theta$, from the origin to the end of the first revolution.

*Ans.* $\pi a\sqrt{1 + 4\pi^2} + \frac{a}{2}\log(2\pi + \sqrt{1 + 4\pi^2})$.

2. Rectify the spiral $\rho = e^{a\theta}$ from the origin to the point $(\rho, \theta)$.

*Hint.* Use ($B$). *Ans.* $\frac{\rho}{a}\sqrt{a^2 + 1}$.

3. Find the entire length of the curve $\rho = a\sin^3\frac{\theta}{3}$. *Ans.* $\frac{3\pi a}{2}$.

4. Find the circumference of the circle $\rho = 2r\sin\theta$. *Ans.* $2\pi r$.

5. Find the length of the hyperbolic spiral $\rho\theta = a$ from $(\rho_1, \theta_1)$ to $(\rho_2, \theta_2)$.

*Ans.* $\sqrt{a^2 + \rho_1^2} - \sqrt{a^2 + \rho_2^2} + a\log\frac{\rho_1(a + \sqrt{a^2 + \rho_2^2})}{\rho_2(a + \sqrt{a^2 + \rho_1^2})}$.

6. Find the length of the arc of the cissoid $\rho = 2a \tan\theta \sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

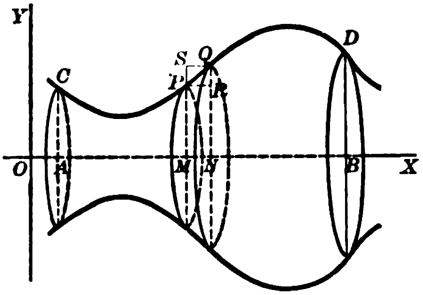
$$\textit{Ans.}\quad 2a\left\{\sqrt{5} - 2 - \sqrt{3}\log\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}(2+\sqrt{3})}\right\}$$

7. Find the length of the parabola $\rho = a\sec^2\frac{\theta}{2}$ from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$$\textit{Ans.}\quad 2a\left(\sec\frac{\pi}{4} + \log\tan\frac{3\pi}{8}\right).$$

8. Show that the entire length of the epicycloid $4(\rho^2 - a^2)^3 = 27a^4\rho^2\sin^2\theta$, which is traced by a point on a circle of radius $\frac{a}{2}$ rolling on a fixed circle of radius $a$, is $12a$.

**227. Volumes of solids of revolution.** Let $V$ denote the volume of the solid generated by revolving the plane surface $AMPC$ about the axis of $X$, the equation of the plane curve $CPD$ being $y = f(x)$.



Let $x (= OM)$ take on a small increment $\Delta x (= MN)$; then $V$ takes on an increment $\Delta V$, the volume of the disc generated by the plane surface $MNQP$. In revolving, the two rectangles $MNRP$ and $MNQS$ generate cylinders having the same altitude $\Delta x (= MN)$, the exterior one having $NQ$, and the interior one $MP$, as radius of the base. The disc generated by $MNPQ$ is evidently greater than the interior but less than the exterior cylinder. Hence

$$\pi\overline{MP}^2\cdot\Delta x < \Delta V < \pi\overline{NQ}^2\cdot\Delta x;$$

or, dividing by $\Delta x$,

$$\pi\overline{MP}^2 < \frac{\Delta V}{\Delta x} < \pi\overline{NQ}^2.$$

Now let $\Delta x$ approach zero as a limit; then $NQ$ approaches $MP$ as a limit, and we get

$$\frac{dV}{dx} = \pi y^2 (= \pi\overline{MP}^2);$$

or, using differentials,

$$dV = \pi y^2 dx,$$

which is the *differential of the volume of the solid of revolution.* Integrating,

$$V = \pi \int y^2 dx.$$

Therefore if $OA = a$ and $OB = b$, the volume generated by revolving $ABDC$ about the axis of $X$ may be calculated by means of the formula

$$(A) \qquad V = \pi \int_a^b y^2 dx,$$

where the value of $y$ in terms of $x$ must be substituted from the equation of the given curve.

This formula is easily recalled if we consider a slice or disc of the solid between two planes perpendicular to the axis of revolution as an element of the volume, and regard it as a cylinder of infinitesimal altitude $dx$ and with a base of area $\pi y^2$, hence of volume $\pi y^2 dx$. Summing up all such slices (elements) from $A$ to $B$, we get the volume generated by revolving $ABDC$ about the axis of $X$.

Similarly when $OY$ is the axis of revolution we use the formula

$$(B) \qquad V = \pi \int_c^d x^2 dy,$$

where the value of $x$ in terms of $y$ must be substituted from the equation of the given curve.

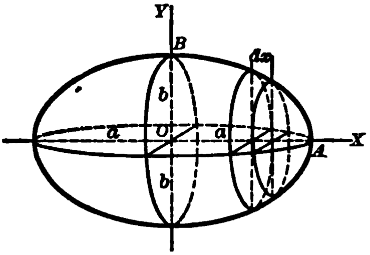
Ex. 1. Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the axis of $X$.

*Solution.* Since $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, and the required volume is twice the volume generated by $OAB$, we get, substituting in $(A)$,

$$\frac{V}{2} = \pi \int_0^a y^2 dx = \pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\, dx$$

$$= \frac{2\pi ab^2}{3}.$$

$$\therefore V = \frac{4\pi ab^2}{3}.$$



To verify this result, let $b = a$. Then $V = \frac{4\pi a^3}{3}$, the volume of a sphere, which is only a special case of the ellipsoid. When the ellipse is revolved about its major axis the solid generated is called a prolate spheroid; when about its minor axis, an oblate spheroid.

## EXAMPLES

1. Find the volume of the sphere generated by revolving the circle $x^2 + y^2 = r^2$ about a diameter. *Ans.* $\frac{4\pi r^3}{3}$.

2. Find by integration the volume of the right cone generated by revolving the line joining the origin to the point $(a, b)$ about the axis of $X$. *Ans.* $\frac{\pi ab^2}{3}$.

3. Find the volume of the cone generated by revolving the line of Ex. 2 about the axis of $Y$. *Ans.* $\frac{\pi a^2 b}{3}$.

4. Find the volume of the paraboloid of revolution generated by revolving the arc of the parabola $y^2 = 4ax$ between the origin and the point $(x_1, y_1)$ about its axis.
*Ans.* $2\pi a x_1^2 = \frac{\pi y_1^2 x_1}{2}$; i.e. one half of the volume of the circumscribing cylinder.

5. Show that the volumes generated by revolving $y = e^x$ about $OX$ and $OY$ are $\frac{\pi}{2}$ and $2\pi$ respectively.

6. Find the volume generated by revolving the arc in Ex. 4 about the axis of $Y$.
*Ans.* $\frac{\pi y_1^5}{80a^2} = \frac{1}{5}\pi x_1^2 y_1$; i.e. one fifth of the cylinder of altitude $y_1$ and radius of base $x_1$.

7. Find by integration the volume of the cone generated by revolving about $OX$ that part of the line $4x - 5y + 3 = 0$ which is intercepted between the coördinate axes. *Ans.* $\frac{9\pi}{100}$.

8. Find the volume of the spindle-shaped solid generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the axis of $X$. *Ans.* $\frac{32\pi a^3}{105}$.

9. Find the volume generated by revolving the witch $y = \frac{8a^3}{x^2 + 4a^2}$ about its asymptote $OX$. *Ans.* $4\pi^2 a^3$.

10. Find the volume generated by revolving about $OY$ that part of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ which is intercepted by the coördinate axis. *Ans.* $\frac{\pi a^3}{15}$.

11. Find the volume of the torus (ring) generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about $OX$. *Ans.* $2\pi^2 a^2 b$.

12. Find the volume generated by revolving one arch of the sine curve $y = \sin x$ about the axis of $X$. *Ans.* $\frac{\pi^2}{2}$.

13. Find the volume generated by revolving about $OX$ the curve

$$(x - 4a)y^2 = ax(x - 3a)$$

between the limits $x = 0$ and $x = 3a$. *Ans.* $\frac{\pi a^3}{2}(15 - 16\log 2)$.

**14.** Find the volume generated by revolving one arch of the cycloid

$$x = r \operatorname{arc\,vers}\frac{y}{r} - \sqrt{2\,ry - y^2}$$

about $OX$, its base.

*Hint.* Substitute $dx = \frac{y\,dy}{\sqrt{2ry - y^2}}$, and limits $y = 0$, $y = 2r$, in (A), p. 385. *Ans.* $5\pi^2 r^3$.

**15.** Find the volume generated by revolving the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ about the axis of $X$ from $x = 0$ to $x = b$.

*Ans.* $\frac{\pi a^3}{4}\left(e^{\frac{2b}{a}} - e^{-\frac{2b}{a}}\right) + \frac{\pi a^2 b}{2}$.

**16.** Find the volume of the solid generated by revolving the cissoid $y^2 = \frac{x^3}{2a - x}$ about its asymptote $x = 2a$.

*Ans.* $2\pi^2 a^3$.

**17.** Given the slope of tangent to the tractrix $\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}$; find the solid generated by revolving it about $OX$.

*Ans.* $\frac{2}{3}\pi a^3$.

**18.** Show that the volume of a conical cap of height $a$ cut from the solid generated by revolving the rectangular hyperbola $x^2 - y^2 = a^2$ about $OX$ equals the volume of a sphere of radius $a$.

**19.** Using the parametric equations of the hypocycloid

$$\begin{cases} x = a\cos^3\theta, \\ y = a\sin^3\theta, \end{cases}$$

find the volume of the solid generated by revolving it about $OX$. *Ans.* $\frac{32\pi a^3}{105}$.

**20.** Find the volume generated by revolving one arch of the cycloid

$$\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta), \end{cases}$$

about its base $OX$. *Ans.* $5\pi^2 a^3$.

Show that if the arch be revolved about $OY$ the volume generated is $6\pi^3 a^3$.

**21.** Show that the volume of the *egg* generated by revolving the curve

$$x^2 y^2 = (x - a)(x - b)$$

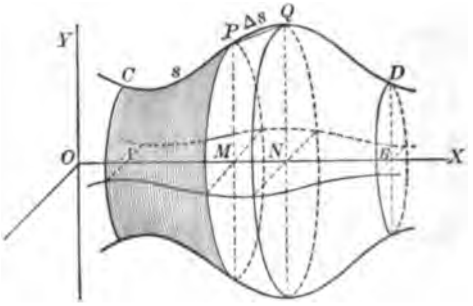
about $OX$ is $$\pi\left\{(a + b)\log\frac{b}{a} - 2(b - a)\right\}.$$

**22.** Find the volume generated by revolving the curve $x^4 - a^2x^2 + a^2y^2 = 0$ about $OX$.

*Ans.* $\frac{4\pi a^3}{15}$.

**23.** Find the volume of the solid generated by revolving the curve $x^2 + y^{\frac{2}{3}} = 1$ about $OY$.

*Ans.* $\frac{4\pi}{5}$.



**228. Areas of surfaces of revolution.** Let $S$ denote the area of the shaded surface generated by revolving the arc $CP (= s)$ about $OX$, the equation of the plane curve $CPD$ being $y = f(x)$.

Let $x (= OM)$ take on a small increment $\Delta x (= MN)$; then $S$ takes on an increment $\Delta S$, the area of the band generated by the arc $PQ (= \Delta s)$. Draw the chord $PQ$. Let $\Delta S'$ denote the area of the convex surface of the frustrum of the cone of revolution generated by the chord $PQ$.

Then

$$\Delta S' = \frac{2\pi y + 2\pi(y + \Delta y)}{2} \cdot \text{chord } PQ,$$

$$\left[\begin{array}{l}\textbf{The lateral area of the frustrum of a cone of revolution is equal to one half}\\ \textbf{the sum of the circumferences of its bases multiplied by the slant height.}\end{array}\right]$$

or,
$$\Delta S' = 2\pi\left(y + \frac{\Delta y}{2}\right) \cdot \text{chord } PQ.$$

Multiplying and dividing the first member by $\Delta S$, and then dividing both members by $\Delta s$, we get

$$(A) \qquad \frac{\Delta S}{\Delta s} \cdot \frac{\Delta S'}{\Delta S} = 2\pi\left(y + \frac{\Delta y}{2}\right) \cdot \frac{\text{chord } PQ}{\Delta s}.$$

Now let $\Delta x$ approach zero as a limit. Then $\Delta s$ and $\Delta y$ also approach zero and

$$\underset{\Delta s = 0}{\text{limit}} \left(\frac{\Delta S}{\Delta s}\right) = \frac{dS}{ds}, \quad \underset{\Delta s = 0}{\text{limit}} \left(\frac{\Delta S'}{\Delta S}\right) = 1, \quad \underset{\Delta s = 0}{\text{limit}} \left(\frac{\text{chord } PQ}{\Delta s}\right) = 1.$$

Hence from $(A)$,

$$\frac{dS}{ds} = 2\pi y \, (= 2\pi MP);$$

or, using differentials,

$$dS = 2\pi y ds,$$

which is the *differential of the area of the surface of revolution.* Integrating,

$$S = 2\pi\int y\,ds;$$

or, since from (29), p. 143, $ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx$, we have

$$S = 2\pi\int y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx.$$

Hence if $OA = a$ and $OB = b$, the surface generated by revolving the arc $CD$ about $OX$ is given by the formula

$$\textbf{(B)}\qquad S = 2\pi\int_a^b y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx,$$

where the value of $y$ and $\frac{dy}{dx}$ in terms of $x$ must be substituted from the equation of the given curve.

This formula is easily remembered if we consider a narrow band of the surface included between two planes perpendicular to the axis of revolution as the element of area, and regard it as the convex surface of a truncated cone of revolution of infinitesimal slant height $ds$ and with a middle section whose circumference equals $2\pi y$, hence of area $2\pi y\,ds$. Summing up all such bands (elements) from $A$ to $B$, we get the area of the surface generated by revolving the arc $CPD$ about $OX$.

Similarly when $OY$ is the axis of revolution we use the formula

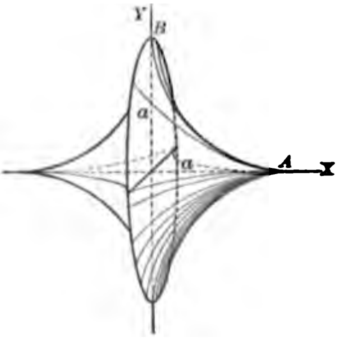
$$\textbf{(C)}\qquad S = 2\pi\int_c^d x\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}dy,$$

where the value of $x$ and $\frac{dy}{dx}$ in terms of $y$ must be substituted from the equation of the given curve.

Ex. 1. Find the area of the surface of revolution generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the axis of $X$.

*Solution.* Here $\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$, $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$.

Substituting in (*B*), p. 389, noting that the arc *BA* generates only one half of the surface, we get



$$\frac{S}{2} = 2\pi\int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}\left[1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right]^{\frac{1}{2}} dx$$

$$= 2\pi\int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}\left(\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{2}} dx$$

$$= 2\pi a^{\frac{1}{3}}\int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} x^{-\frac{1}{3}} dx$$

$$= \frac{6\pi a^2}{5}.$$

$$\therefore S = \frac{12\pi a^2}{5}.$$

## EXAMPLES

1. Find the area of the surface of the sphere generated by revolving the circle $x^2 + y^2 = r^2$ about a diameter. *Ans.* $4\pi r^2$.

2. Find the area of the surface generated by revolving the parabola $y^2 = 4ax$ about $OX$, from the origin to the point where $x = 3a$. *Ans.* $\frac{56}{3}\pi a^2$.

3. Find by integration the area of the surface of the cone generated by revolving the line joining the origin to the point $(a, b)$ about $OX$. *Ans.* $\pi b\sqrt{a^2 + b^2}$.

4. Find the surface generated by revolving the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ about $OY$ from $x = 0$ to $x = a$. *Ans.* $2\pi a^2(1 - e^{-1})$.

5. Find the surface of the prolate spheroid generated by revolving the ellipse $y^2 = (1 - e^2)(a^2 - x^2)$ about $OX$. *Ans.* $2\pi b^2 + \frac{2\pi ab}{e}\text{arc sin}\, e$.

6. Find the surface of the torus (ring) generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about $OX$. *Ans.* $4\pi^2 ab$.

*Hint.* Using the positive value of $\sqrt{a^2 - x^2}$ gives the outside surface, and the negative value the inside surface.

7. Find the surface generated by revolving an arch of the cycloid

$$x = r\,\text{arc vers}\,\frac{y}{r} - \sqrt{2ry - y^2}$$

about its base. *Ans.* $\frac{64\pi r^2}{3}$.

8. Find the surface generated by the cycloid when revolved about the tangent at its vertex. *Ans.* $\frac{32\pi r^2}{3}$.

9. Find the surface of the oblate spheroid generated by revolving the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ about its minor axis. *Ans.* $2\pi a^2 + \pi\frac{b^2}{e}\log\frac{1+e}{1-e}$.

*Hint.* $e$ = eccentricity of ellipse.

**10.** Find the surface generated when the cycloid is revolved about its axis.

*Ans.* $8\pi r^2(\pi - \frac{4}{3})$.

**11.** Find the surface generated by revolving about $OX$ that portion of the curve $y = e^x$ which is to the left of the axis of $Y$. *Ans.* $\pi[\sqrt{2} + \log(1 + \sqrt{2})]$.

**12.** A quadrant of a circle of radius $a$ revolves about the tangent at one extremity; prove that the area of the curved surface generated is $\pi(\pi - 2)a^2$.

**13.** Find the surface generated by revolving the cardioid

$$\begin{cases} x = a(2\cos\theta - \cos 2\theta), \\ y = a(2\sin\theta - \sin 2\theta), \end{cases}$$

about $OX$. *Ans.* $\frac{128\,a^2\pi}{5}$.

**14.** Show that the surface generated by revolving the cycloid

$$\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta), \end{cases}$$

about $OX$ is $\frac{64\,\pi a^2}{3}$; and that when revolved about $OY$ it is $\frac{32\,\pi a^2}{3}$.

**15.** Show that the surface generated by revolving the curve $x^4 - a^2x^2 + 8\,a^2y^2 = 0$ about $OX$ from $x = 0$ to $x = a$ is $\frac{\pi}{4\,a^2}\int_0^a (3\,a^2x - 2\,x^3)\,dx = \frac{1}{4}\pi a^2$.

**16.** Show that the surface generated by revolving $x^4 + 3 = 6\,xy$ about $OY$ from $x = 1$ to $x = 2$ is $\pi\left(\frac{15}{4} + \log 2\right)$; and that when revolved about $OX$ it is $\frac{47\,\pi}{16}$.

**17.** Show that the surface generated by revolving the cubical parabola $y = x^3$ about $OX$ from $x = 0$ to $x = 1$ is $2\,\pi\int_0^1 \sqrt{1 + 9\,x^4}\,x^3dx = \frac{\pi}{16}(\sqrt{1000} - 1)$.

**18.** Show that if we rotate $y^2 + 4\,x = 2\log y$ about $OX$ from $y = 1$ to $y = 2$, the surface generated is $\frac{10\,\pi}{3}$.

# CHAPTER XXXI

## SUCCESSIVE AND PARTIAL INTEGRATION

**229. Successive Integration.** Corresponding to *successive differentiation* in the Differential Calculus we have the inverse process of *successive integration* in the Integral Calculus. We shall illustrate by means of examples the details of this process, and show how problems arise where it is necessary to apply it.

Ex. 1. Given $\frac{d^3y}{dx^3} = 6x$; to find $y$.

*Solution.* We may write this

$$\frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = 6x,$$

or,

$$d\left(\frac{d^2y}{dx^2}\right) = 6\,xdx.$$

Integrating,

$$\frac{d^2y}{dx^2} = \int 6\,xdx,$$

or,

$$\frac{d^2y}{dx^2} = 3\,x^2 + c_1.$$

This may also be written

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = 3\,x^2 + c_1,$$

or,

$$d\left(\frac{dy}{dx}\right) = (3\,x^2 + c_1)\,dx.$$

Integrating again,

$$\frac{dy}{dx} = \int (3\,x^2 + c_1)\,dx, \text{ or,}$$

(*A*)

$$\frac{dy}{dx} = x^3 + c_1x + c_2.$$

Again,

$$dy = (x^3 + c_1x + c_2)\,dx, \text{ and integrating,}$$

(*B*)

$$y = \frac{x^4}{4} + \frac{c_1x^2}{2} + c_2x + c_3. \quad \textit{Ans.}$$

The result (*A*) is also written in the form

$$\frac{dy}{dx} = \int\int 6\,xdxdx \ \left(\text{or } = \int\int 6\,xdx^2\right),$$

and is called a *double integral*, while (*B*) is written in the form

$$y = \int\int\int 6\,x dx dx dx \quad (\text{or} = \int\int\int 6\,x dx^3),$$

and is called a *triple integral*. In general, a *multiple integral* requires two or more successive integrations. As before, if there are no limits assigned, as in the above example, the integral is indefinite; if there are limits assigned for each successive integration, the integral is definite.

**Ex. 2.** Find the equation of a curve for every point of which the second derivative of the ordinate with respect to the abscissa equals 4.

*Solution.* Here $\frac{d^2y}{dx^2} = 4$. Integrating as in Ex. 1,

(*C*) $\frac{dy}{dx} = 4x + c_1$.

(*D*) $y = 2x^2 + c_1x + c_2$. *Ans.*

This is the equation of a parabola with its axis parallel to $OY$ and extending upward. By giving the arbitrary constants of integration $c_1$ and $c_2$ all possible values we obtain all such parabolas.

In order to determine $c_1$ and $c_2$, two more conditions are necessary. Suppose we say (a) that at the point where $x = 2$ the slope of the tangent to the parabola is zero; and (b) that the parabola passes through the point $(2, -1)$.

(a) Substituting $x = 2$ and $\frac{dy}{dx} = 0$ in (*C*)

gives $0 = 8 + c_1$.

Hence $c_1 = -8$,

and (*D*) becomes $y = 2x^2 - 8x + c_2$.

(b) The coördinates of $(2, -1)$ must satisfy this equation; therefore

$$-1 = 8 - 16 + c_2, \text{ or, } c_2 = -7.$$

Therefore the equation of the particular parabola which satisfies all three conditions is $y = 2x^2 - 8x - 7$.

## EXAMPLES

1. Given $\frac{d^3y}{dx^3} = ax^2$; find $y$. *Ans.* $y = \frac{ax^5}{60} + \frac{c_1x^2}{2} + c_2x + c_3$.

2. Given $\frac{d^3y}{dx^3} = 0$; find $y$. *Ans.* $y = \frac{c_1x^2}{2} + c_2x + c_3$.

3. Given $d^3y = \frac{2\,dx^3}{x^3}$; find $y$. *Ans.* $y = \log x + \frac{c_1x^2}{2} + c_2x + c_3$.

4. Given $\frac{d^3\rho}{d\theta^3} = \sin\theta$; find $\rho$. *Ans.* $\rho = \cos\theta + \frac{c_1\theta^2}{2} + c_2\theta + c_3$.

5. Given $\frac{d^3s}{dt^3} = 3t^2 - \frac{1}{t^3}$; find $s$. *Ans.* $s = \frac{t^5}{20} + \frac{1}{2}\log t + \frac{c_1t^2}{2} + c_2t + c_3$

6. Given $d^2\rho = \sin\phi\cos^2\phi\, d\phi^2$; find $\rho$. *Ans.* $\rho = \frac{\sin^3\phi}{6} + c_1\sin\phi + c_2$

7. Determine the equations of all curves having zero curvature.

*Hint.* $\frac{d^2y}{dx^2} = 0$, from (38), p. 161, since $K = 0$.

*Ans.* $y = c_1x + c_2$, a doubly infinite system of straight lines.

8. The acceleration of a moving point is constant and equal to $f$; find the distance (space) traversed.

*Hint.* From Ex. 28, p. 114, $\frac{d^2s}{dt^2} = f$. *Ans.* $s = \frac{ft^2}{2} + c_1t + c_2$.

9. Show in Ex. 8 that $c_1$ stands for the initial velocity and $c_2$ for the initial distance.

10. Find the equation of the curve at each point of which the second derivative of the ordinate with respect to the abscissa is four times the abscissa, and which passes through the origin and the point (2, 4). *Ans.* $3y = 2x(x^2 - 1)$.

11. Given $\frac{d^4y}{dx^4} = x\cos x$; find $y$. *Ans.* $y = x\cos x - 4\sin x + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4$.

12. Given $\frac{d^3y}{dx^3} = \sin^3 x$; find $y$. *Ans.* $y = \frac{7\cos x}{9} - \frac{\cos^3 x}{27} + \frac{c_1x^2}{2} + c_2x + c_3$.

**230. Partial Integration.** Corresponding to *partial differentiation* in the Differential Calculus we have the inverse process of *partial integration* in the Integral Calculus. As may be inferred from the connection, partial integration means that, having given a differential expression involving two or more independent variables, we integrate it, considering first *a single one only* as varying and all the rest constant. Then we integrate the result, considering another one as varying and the others constant, and so on. Such integrals are called *double*, *triple*, etc., according to the number of variables, and are called *multiple integrals.*[*]

Thus the expression

$$u = \int\int f(x, y)\,dydx$$

indicates that we wish to find a function $u$ of $x$ and $y$ such that

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y).$$

---

[*] The integrals of the same name in the last section are special cases of these, namely, when we integrate with respect to the same variable throughout.

In the solution of this problem the only new feature is that the constant of integration has a new form. We shall illustrate this by means of examples. Thus, suppose we wish to find $u$, having given

$$\frac{\partial u}{\partial x} = 2x + y + 3.$$

Integrating this with respect to $x$, considering $y$ as constant, we have

$$u = x^2 + xy + 3x + \phi,$$

where $\phi$ denotes the constant of integration. But since $y$ was regarded as constant during this integration, it may happen that $\phi$ involves $y$ in some way; in fact $\phi$ will in general be a function of $y$. We shall then indicate this dependence of $\phi$ on $y$ by replacing $\phi$ by the symbol $\phi(y)$. Hence the most general form of $u$ is

$$u = x^2 + xy + 3x + \phi(y),$$

where $\phi(y)$ *denotes an arbitrary function of* $y$.

As another problem let us find

$$(A) \qquad u = \int\int (x^2 + y^2)\, dy dx.$$

This means that we wish to find $u$, having given

$$\frac{\partial^2 u}{\partial x \partial y} = x^2 + y^2.$$

Integrating first with respect to $y$, regarding $x$ as constant, we get

$$\frac{\partial u}{\partial x} = x^2 y + \frac{y^3}{3} + \psi(x),$$

where $\psi(x)$ is an arbitrary function of $x$ and is to be regarded as the constant of integration.

Now integrating this result with respect to $x$, regarding $y$ as constant, we have

$$u = \frac{x^3 y}{3} + \frac{xy^3}{3} + \Psi(x) + \Phi(y),$$

where $\Phi(y)$ is the constant of integration, and

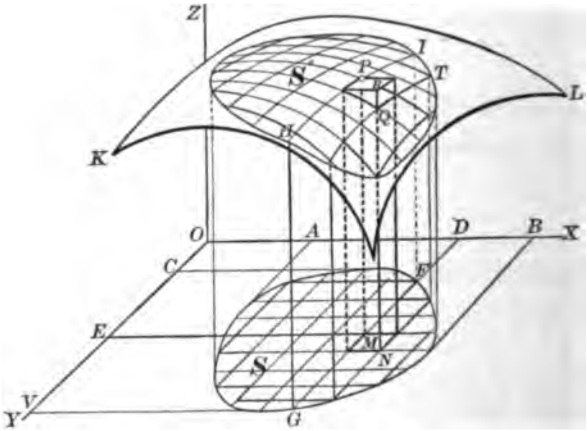
$$\Psi(x) = \int \psi(x) dx.$$

**231. Definite double integral. Geometric interpretation.** Let $f(x, y)$ be a continuous and single-valued function of $x$ and $y$. Geometrically,

$$(A) \qquad z = f(x, y)$$

is the equation of a surface, as $KL$. Take some area $S$ in the $XY$ plane and construct upon $S$ as a base, the right cylinder whose elements are accordingly parallel to $OZ$. Let this cylinder intersect $KL$ in the area $S'$, and now let us find the volume $V$ of the solid bounded by $S$, $S'$, and the cylindrical surface. We proceed as follows.

At equal distances apart ($= \Delta x$) in the area $S$ draw a set of lines parallel to $OY$, and then a second set parallel to $OX$ at equal distances apart ($= \Delta y$). Through these lines pass planes parallel to $YOZ$ and $XOZ$ respectively. Then within the areas $S$ and $S'$ we have a network of lines, as in the figure, that in $S$ being composed of rectangles, each of area $\Delta x \cdot \Delta y$. This construction divides the cylinder into a number of vertical columns, such as $MNPQ$, whose upper and lower bases are corresponding portions of the networks in $S'$ and $S$ respectively. As the upper bases of these columns are curvilinear, we of course cannot calculate the volume of the columns directly. Let us replace these columns by prisms whose upper bases are found thus: each column is cut through by a plane parallel to $XY$ passed through that vertex



of the upper base for which $x$ and $y$ have the least numerical values. Thus the column $MNPQ$ is replaced by the right prism $MNPR$, the upper base being in a plane through $P$ parallel to the $XOY$ plane.

If the coördinates of $P$ are $(x, y, z)$, then $MP = z = f(x, y)$, and therefore

$$(B) \qquad \text{Volume of } MNPR = f(x, y)\Delta y \cdot \Delta x.$$

Calculating the volume of each of the other prisms formed in the same way by replacing $x$ and $y$ in $(B)$ by corresponding values, and adding the results, we obtain a volume $V'$ approximately equal to $V$; that is,

$$(C) \qquad V' = \sum\sum f(x, y)\Delta y \cdot \Delta x;$$

where the double summation sign $\sum\sum$ indicates that there are *two variables* in the quantity to be summed up.

If now in the figure we increase the number of divisions of the network in $S$ indefinitely by letting $\Delta x$ and $\Delta y$ diminish indefinitely, and calculate in each case the double sum $(C)$, then obviously $V'$ will approach $V$ as a limit, and hence we have the fundamental result

$$(D) \qquad V = \underset{\substack{\Delta y = 0\\ \Delta x = 0}}{\text{limit}} \sum\sum f(x, y)\Delta y \cdot \Delta x.$$

Let us see how to calculate this double limit. We commence by calculating $(C)$ for all the prisms of a row parallel to $YOZ$, say along the line $DG$.

This will give us, approximately of course, the volume of a slice of $V$ bounded by planes through $P$ and $Q$ parallel to the $YOZ$ plane. To do this analytically, we sum up in $(C)$, keeping $x$ constant $(= OD)$. Since in this summation $\Delta x$ is also constant, we may write $(C)$ in the form

$$(E) \qquad V' = \sum \Delta x \cdot \sum f(x, y)\Delta y.$$

Hence $(D)$ becomes

$$(F) \qquad V = \underset{\substack{\Delta y = 0\\ \Delta x = 0}}{\text{limit}} \sum \Delta x \cdot \sum f(x, y)\Delta y.$$

In $(E)$, the limits for the second sign of summation are the extreme values of $y$ for the vertices of the network along the line $DG$, and for the first sign of summation the extreme values of $x$ in the entire network. Hence it should now be *intuitionally* evident that $(F)$ becomes*

$$(G) \qquad V = \int_{OA}^{OB} dx \int_{DF}^{DG} f(x, y)\, dy,$$

for we have merely replaced the signs of summation by integral signs, and the limits by the values taken from the region $S$ itself. We have accordingly the fundamental result,

$$(H) \quad V = \underset{\substack{\Delta y = 0 \\ \Delta x = 0}}{\text{limit}} \sum\sum f(x, y)\, \Delta y \cdot \Delta x = \int_{OA}^{OB} \int_{DF}^{DG} f(x, y)\, dy dx,$$

the second integration sign applying to $y$ and being performed first, $x$ and $dx$ being meanwhile regarded as constants.

The process of evaluating $(D)$ might have been carried out by first adding up the columns in $(C)$ along a line parallel to $OX$, i.e. $y$ remaining constant, afterwards summing up the resulting prisms by varying $y$, and finally passing to the limit as $\Delta x$ and $\Delta y$ approach zero. These steps would be indicated by writing the differential expression in $(F)$ in the form

$$f(x, y)\, dx dy$$

and changing the limits. Summing up our line of reasoning, we may write

$$V = \underset{\substack{\Delta y = 0 \\ \Delta x = 0}}{\text{limit}} \sum\sum f(x, y)\, \Delta y \cdot \Delta x = \int_{a_2}^{a_1} \int_{u_2}^{u_1} f(x, y)\, dy dx$$

$$= \int_{b_2}^{b_1} \int_{v_2}^{v_1} f(x, y)\, dx dy,$$

where $v_1$ and $v_2$ are in general functions of $y$, and $u_1$ and $u_2$ functions of $x$, the second integral sign applying to the first differential and being calculated first.

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* A rigorous proof of $(G)$ is to be found in Goursat's *Cours d'Analyse Mathématique*, Vol. I, § 123. An English translation, by Professor Hedrick, of this book is published by Ginn & Company.

Our result may be stated in the following form:

*The definite double integral*

$$\int_{a_2}^{a_1}\int_{u_2}^{u_1} f(x, y)\,dydx$$

*may be interpreted as that portion of the volume of a truncated right cylinder which is included between the plane XOY and the surface*

$$z = f(x, y),$$

*the base of the cylinder being the area bounded by the curves*

$$y = u_1, \quad y = u_2, \quad x = a_1, \quad x = a_2.$$

Similarly for the second integral.

It is instructive to look upon the above process of finding the volume of the solid as follows:

Consider a column of infinitesimal base $dydx$ and altitude $z$ as an element of the volume. Summing up all such elements from $y = DF$ to $y = DG$, $x$ in the meanwhile being constant (say $= OD$), gives the volume of an indefinitely thin slice having $FGHI$ as one face. The volume of the whole solid is then found by summing up all such slices from $x = OA$ to $x = OB$.

In partial integration involving two variables the order of integration denotes that the limits on the inside integral sign correspond to the variable whose differential is written inside, the differentials of the variables and their corresponding limits on the integral signs being written in the reverse order.

**Ex. 1. Find the value of the definite double integral**

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} (x + y)\,dydx.$$

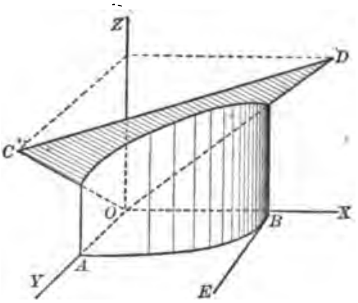
*Solution.* $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} (x+y)\,dydx$

$$= \int_0^a\left[\int_0^{\sqrt{a^2-x^2}} (x+y)\,dy\right]dx$$

$$= \int_0^a\left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x}} dx$$

$$= \int_0^a\left(x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right)dx$$

$$= \frac{2a^3}{3}. \quad \textit{Ans.}$$



Interpreting this result geometrically it means that we have found the volume of the solid of cylindrical shape standing on $OAB$ as base and bounded at the top by the surface (plane) $z = x + y$.

The attention of the student is now particularly called to the manner in which the limits do bound the base $OAB$, which corresponds to the area $S$ in the figure p. 396. Our solid here stands on a base in the $XY$ plane bounded by

$$\left.\begin{array}{l} y = 0 \text{ (line } OB) \\ y = \sqrt{a^2 - x^2} \text{ (quadrant of circle } AB) \end{array}\right\} \text{from } y \text{ limits;}$$

$$\left.\begin{array}{l} x = 0 \text{ (line } OA) \\ x = a \text{ (line } BE) \end{array}\right\} \text{from } x \text{ limits.}$$

**232. Value of a definite double integral over a region $S$.** In the last section we represented the definite double integral as a volume. This does not necessarily mean that every definite double integral is a volume, for the physical interpretation of the result depends on the nature of the quantities represented by $x$, $y$, $z$. Thus, if $x$, $y$, $z$ are simply considered as the coördinates of a point in space, and nothing more, then the result is indeed a volume. In order to give the definite double integral in question an interpretation not necessarily involving the geometrical concept of volume, we observe at once that the variable $z$ does not occur explicitly in the integral, and therefore we may confine ourselves to the $XY$ plane. In fact, let us consider simply a region $S$ in the $XY$ plane, and a given function $f(x, y)$. Then, drawing a network as before, calculate the value of

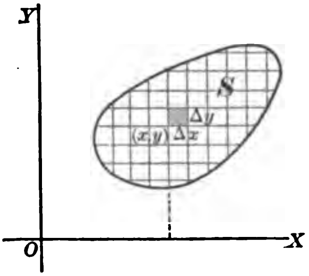
$$f(x, y)\,\Delta y \Delta x$$

*for each point of the network*, and sum up, finding in this way

$$\sum\sum f(x, y)\,\Delta y \Delta x,$$

and finally pass to the limit as $\Delta x$ and $\Delta y$ approach zero. This operation we call *integrating the function $f(x, y)$ over the region $S$*, and it is denoted by the symbol

$$\iint_S f(x, y)\,dydx.$$



If $S$ is bounded by the curves $x = a_1$, $x = a_2$, $y = u_1$, $y = u_2$, then

$$\iint_S f(x, y)\,dydx = \int_{a_1}^{a_2}\int_{u_1}^{u_2} f(x, y)\,dydx.$$

We may state our result as follows:

*To integrate a given function $f(x, y)$ over a given region $S$ in the $XOY$ plane means to calculate the value of*

$$\lim_{\substack{\Delta x = 0 \\ \Delta y = 0}} \sum\sum f(x, y)\,\Delta y \Delta x,$$

*as explained above, and the result is equal to the definite double integral*

$$\int_{a_1}^{a_2}\int_{u_1}^{u_2} f(x, y)\,dydx, \text{ or } \int_{b_1}^{b_2}\int_{v_1}^{v_2} f(x, y)\,dxdy,$$

*the limits being chosen so that the entire region $S$ is covered. This process is indicated briefly by*

$$\iint\limits_S f(x, y)\,dydx.$$

In the sections which follow we shall show by numerous examples how the area of the region itself may be calculated in this way, and also the moment of inertia of the region.

Before attempting to apply partial integration to practical problems it is best that the student should acquire by practice some facility in evaluating definite multiple integrals.

Ex. 1. Verify $\int_b^{2b}\int_0^a (a - y)\,x^2dydx = \frac{7a^2b^3}{6}$.

*Solution.* $\int_b^{2b}\int_0^a (a - y)\,x^2dydx = \int_b^{2b}\left[ay - \frac{y^2}{2}\right]_0^a x^2dx = \int_b^{2b}\frac{a^2}{2}x^2dx = \frac{7a^2b^3}{6}.$

Ex. 2. Verify $\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} xdydx = \frac{2a^3}{3}$.

*Solution.*

$$\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} xdydx = \int_0^a\left[xy\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a 2x\sqrt{a^2 - x^2}\,dx = \left[-\frac{2}{3}\left(a^2 - x^2\right)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}a^3.$$

In partial integration involving three variables the order of integration is denoted in the same way as for two variables; that is, the order of the limits on the integral signs, reading from the inside to the left, is the same as the order of the corresponding variables whose differentials are read from the inside to the right.

**Ex. 3.** Verify $\int_2^3\int_1^2\int_2^5 xy^2dzdydx = \frac{35}{2}$.

*Solution.*
$$\int_2^3\int_1^2\int_2^5 xy^2dzdydx = \int_2^3\int_1^2\left[\int_2^5 xy^2dz\right]dydx = \int_2^3\int_1^2\left[xy^2z\right]_2^5 dydx$$
$$= 3\int_2^3\int_1^2 xy^2dydx = \int_2^3\left[\int_1^2 xy^2dy\right]dx$$
$$= 3\int_2^3\left[\frac{xy^3}{3}\right]_1^2 dx = 7\int_2^3 xdx = \frac{35}{2}.$$

### EXAMPLES

Verify the following.

1. $\int_0^a\int_0^b xy(x-y)\,dydx = \frac{a^2b^2}{6}(a-b)$.

3. $\int_0^a\int_0^{\sqrt{x}} dydx = \frac{2}{3}a^{\frac{3}{2}}$.

2. $\int_{\frac{b}{2}}^b\int_0^{\frac{r}{b}} rd\theta dr = \frac{7\,b^2}{24}$.

4. $\int_0^a\int_{2x}^{x^{\frac{3}{2}}} xy^2dydx = \frac{2}{3}a^5\left(\frac{a^{\frac{3}{2}}}{13}-\frac{4}{5}\right)$.

5. $\int_b^a\int_\beta^\alpha r^2\sin\theta d\theta dr = \frac{a^3-b^3}{3}(\cos\beta-\cos\alpha)$.

6. $\int_0^a\int_{y-a}^{2y} xydxdy = \frac{11\,a^4}{24}$.

8. $\int_0^a\int_0^x\int_0^y x^3y^2zdzdydx = \frac{a^9}{90}$.

7. $\int_0^a\int_0^b\int_a^{2a} x^2y^2zdzdydx = \frac{a^2b^3}{6}(a^3-b^3)$.

9. $2\,a\int_0^a\int_0^{\sqrt{a^2-ax}}\frac{dzdx}{\sqrt{ax-x^2}} = 4\,a^3$.

10. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\frac{x^2+y^2}{a}} dzdydx = \frac{3\,\pi a^3}{4}$.

11. $\int_0^\pi\int_0^{a(1+\cos\theta)} r^2\sin\theta drd\theta = \frac{4\,a^3}{3}$.

12. $\int_0^b\int_t^{10\,t}\sqrt{st-t^2}\,dsdt = 6\,b^3$.

13. $\int_a^{2a}\int_v^{\frac{v^2}{a}}(w+2\,v)\,dwdv = \frac{143\,a^3}{30}$.

14. $\int_0^1\int_0^x\int_0^{x+y} e^{x+y+z}dzdydx = \frac{e^4-3}{8}-\frac{3\,e^2}{4}+e$.

**233. Plane area as a definite double integral. Rectangular coördinates.** As a simple application of the theorem of the last section (p. 401), we shall now determine the area of the region $S$ itself in the $XOY$ plane by double integration.*

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* Some of the examples that will be given in this and the following articles may be solved by means of a single integration by methods already explained (§§ 221, 223). The only reason in such cases for using successive integration is to familiarize the student with a new method for solution which is sometimes the only one possible.

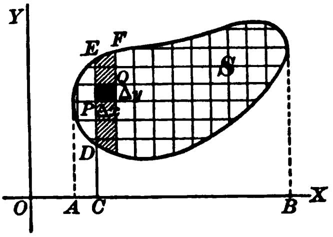
As before, draw lines parallel to $OX$ and $OY$ at distances $\Delta x$ and $\Delta y$ respectively. Now take

$$\text{element of area} = \text{area of rectangle } PQ = \Delta y \cdot \Delta x,$$

the coördinates of $P$ being $(x, y)$.

Denoting by $A$ the entire area of region $S$, we have

$$(A) \qquad A = \underset{\substack{\Delta x = 0 \\ \Delta y = 0}}{\text{limit}} \sum\sum \Delta y \cdot \Delta x.$$



We calculate this by the theorem on p. 401, setting $f(x, y) = 1$, and get

$$(B) \qquad A = \int_{OA}^{OB}\int_{CD}^{CE} dydx,$$

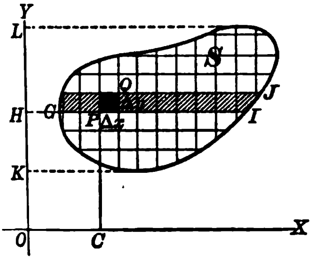
where $CD$ and $CE$ are in general functions of $x$, and $OA$ and $OB$ are constants giving the extreme values of $x$, all four of these quantities being determined from the equations of the curve or curves which bound the region $S$.

It is instructive to interpret this double integral geometrically by referring to our figure. When we integrate first with respect to $y$, keeping $x$ $(= OC)$ constant, we are summing up all the elements in a vertical strip (as $DF$). Then integrating the result with respect to $x$ means that we are summing up all such vertical strips included in the region, and this obviously gives the entire area of the region $S$.

Or, if we change the order of integration, we have

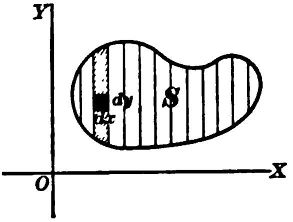
$$(C) \qquad A = \int_{OK}^{OL}\int_{HG}^{HI} dxdy,$$

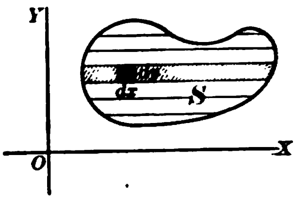
where $HG$ and $HI$ are in general functions of $y$, and $OK$ and $OL$ are constants giving the extreme values of $y$, all four of these quantities being determined from the equations of the curve or curves which bound the region $S$. Geometrically this means that we now commence by summing up all the elements in a horizontal strip (as $GJ$), and then find the entire area by summing up all such strips within the region.



Corresponding to the two orders of summation (integration), the following notation and figures are sometimes used:

$$(D) \qquad A = \int\!\!\int_S dydx; \quad A = \int\!\!\int_S dxdy.$$





Referring to the result stated on page 401, we may say:

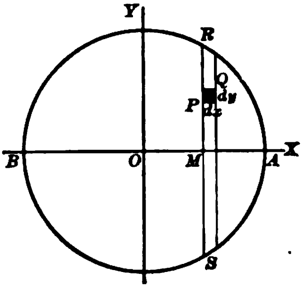
*The area of any region is the value of the double integral of the function $f(x, y) = 1$ taken over that region.*

Or, also, from § 231, p. 399,

*The area equals numerically the volume of a right prism of unit height erected on the base S.*

Ex. 1. Calculate the area of the circle $x^2 + y^2 = r^2$ by double integration.

*Solution.* Summing up first the elements in a vertical strip, we have from (*B*), p. 403,



$$A = \int_{OB}^{OA}\int_{MS}^{MR} dydx.$$

From the equation of the boundary curve (circle) we get

$$MR = \sqrt{r^2 - x^2},\ MS = -\sqrt{r^2 - x^2},$$
$$OB = -r,\ OA = r.$$

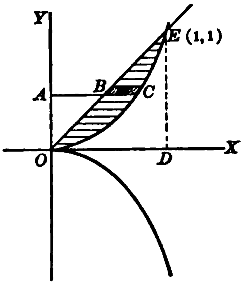
Hence
$$A = \int_{-r}^{+r}\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}} dydx$$
$$= 2\int_{-r}^{r}\sqrt{r^2 - x^2}dx = \pi r^2. \quad \textit{Ans.}$$

When the region whose area we wish to find is symmetrical with respect to one or both of the coördinate axes, it sometimes saves us labor to calculate the area of only a part at first. In the above example we may choose our limits so as to cover only one quadrant of the circle, and then multiply the result by 4. Thus,

$$\frac{A}{4} = \int_0^r\int_0^{\sqrt{r^2-x^2}} dydx = \int_0^r \sqrt{r^2 - x^2}dx = \frac{\pi r^2}{4}.$$

$$\therefore A = \pi r^2. \quad \textit{Ans.}$$

**Ex. 2.** Calculate that portion of the area which lies above $OX$ bounded by he semicubical parabola $y^2 = x^3$ and the straight line $y = x$.



*Solution.* Summing up first the elements in a horizontal strip, we have from $(C)$, p. 403,

$$A = \int_0^{OD}\int_{AB}^{AC} dxdy.$$

From the equation of the line, $AB = y$, and from the equation of the curve, $AC = y^{\frac{2}{3}}$, solving each one for $x$. To determine $OD$, solve the two equations simultaneously to find the point of intersection $E$. This gives the point $(1, 1)$; hence $OD = 1$. Therefore

$$A = \int_0^1\int_y^{y^{\frac{2}{3}}} dxdy = \int_0^1 (y^{\frac{2}{3}} - y)\,dy = \left[\frac{3y^{\frac{5}{3}}}{5} - \frac{y^2}{2}\right]_0^1$$
$$= \tfrac{3}{5} - \tfrac{1}{2} = \tfrac{1}{10}. \quad \textit{Ans.}$$

## EXAMPLES

1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

*Ans.* $4\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx = \pi ab.$

2. Find by double integration the area between the straight line and a parabola with its axis along $OX$, each of which joins the origin and the point $(a, b)$.

*Ans.* $\int_0^a\int_{\frac{bx}{a}}^{b\sqrt{\frac{x}{a}}} dydx = \frac{ab}{6}.$

3. Find by double integration the area of the rectangle formed by the coördinate axes and two lines through $(a, b)$ parallel to the coördinate axes. *Ans.* $ab$.

4. Find by double integration the area of the triangle formed by $OX$ and the lines $x = a$ and $y = \frac{b}{a}x$. *Ans.* $\frac{ab}{2}$.

5. Find by double integration the area between the two parabolas $3y^2 = 25x$ and $5x^2 = 9y$. *Ans.* 5.

6. Required the area in the first quadrant which lies between the parabola $y^2 = ax$ and the circle $y^2 = 2ax - x^2$. *Ans.* $\frac{\pi a^2}{4} - \frac{2a^2}{3}$.

7. Find the area outside of the parabola $y^2 = 4a(a - x)$ and inside of the circle $y^2 = 4a^2 - x^2$.

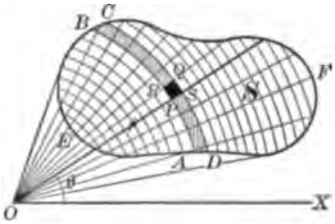
**8.** Solve the problems on page 405 by first summing up all the elements in a horizontal strip, and then summing up all such strips.

*Ans.* Ex. 3, $\int_0^b\int_0^a dxdy = ab.$ Ex. 4, $\int_0^b\int_0^{\frac{ay}{b}} dxdy = \frac{ab}{2}.$

Ex. 5, $\int_0^5\int_{\frac{3y}{25}}^{3\sqrt{\frac{y}{5}}} dxdy = 5.$ Ex. 6, $\int_0^a\int_{a-\sqrt{a^2-x^2}}^{\frac{y^2}{a}} dxdy = \frac{\pi a^2}{4} - \frac{2a^2}{3}.$

## 234. Plane area as a definite double integral. Polar coördinates.

Suppose the equations of the curve or curves which bound the region $S$ are given in polar coördinates. Then the region may be divided into checks bounded by radial lines drawn from the origin, each one making the angle $\Delta\theta$ with the *next* one, and concentric circles drawn with centers at the origin, the difference between each radius and the next one being $\Delta\rho$.



We shall consider these checks as the elements of area of the region $S$. Let us calculate the area of one, say $PQ$, bounded by arcs with radii $\rho$ and $\rho + \Delta\rho$.

From Geometry,

$$\text{area of sector } OSQ = \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta,$$
$$\text{area of sector } OPR = \tfrac{1}{2}\rho^2\Delta\theta.$$

Hence

$$\text{area of element } PQ = \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta - \tfrac{1}{2}\rho^2\Delta\theta$$
$$= \rho\Delta\rho\Delta\theta + \tfrac{1}{2}\overline{\Delta\rho}^2\cdot\Delta\theta.$$

Then, as before, the area of the region $S$ will be

$$A = \underset{\substack{\Delta\rho = 0\\ \Delta\theta = 0}}{\text{limit}}\sum\sum(\rho\Delta\rho\Delta\theta + \tfrac{1}{2}\overline{\Delta\rho}^2\Delta\theta)$$
$$= \int\int\rho d\rho d\theta + \underset{\substack{\Delta\rho = 0\\ \Delta\theta = 0}}{\text{limit}}\Delta\rho\cdot\tfrac{1}{2}\sum\sum d\rho d\theta,$$

the summation extending over the entire region, or,

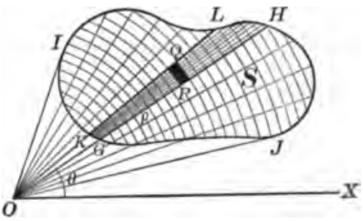
$$(A) \qquad A = \underset{S}{\int\int}\rho d\rho d\theta.$$

Here, again, the summation (integration) may be effected in two ways.

When we integrate first with respect to $\theta$, keeping $\rho$ constant, it means that we sum up all the elements (checks) in a segment of a circular ring (as *ABCD*), and next integrating with respect to $\rho$, that we sum up all such rings within the entire region. Our limits then appear as follows:

$$(B)\qquad A=\int_{OE}^{OF}\int_{\text{angle }XOA}^{\text{angle }XOB}\rho d\theta d\rho,$$

the angles *XOA* and *XOB* being in general functions of $\rho$, and *OE* and *OF* constants giving the extreme values of $\rho$.



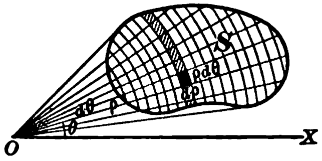
Suppose we now reverse the order of integration. Integrating first with respect to $\rho$, keeping $\theta$ constant, means that we sum up all the elements (checks) in a wedge-shaped strip (as *GKLH*). Then integrating with respect to $\theta$, we sum up all such strips within the region *S*. Here

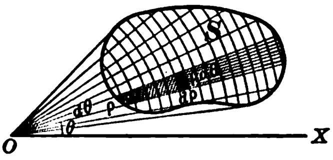
$$(C)\qquad A=\int_{\text{angle }XOJ}^{\text{angle }XOI}\int_{OG}^{OH}\rho d\rho d\theta,$$

*OH* and *OG* being in general functions of $\theta$, and the angles *XOJ* and *XOI* being constants giving the extreme values of $\theta$.

Corresponding to the two orders of summation (integration), the following notation and figures may be conveniently employed:

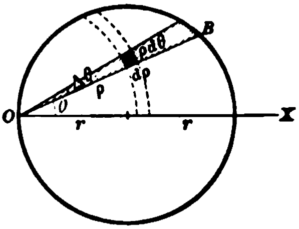
$$(D)\qquad A=\iint\limits_{S}\rho d\theta d\rho,\quad A=\iint\limits_{S}\rho d\rho d\theta.$$





These are easily remembered if we think of the elements (checks) as being rectangles with dimensions $\rho d\theta$ and $d\rho$, and hence of area $\rho d\theta d\rho$.

**Ex. 1.** Find the area of the circle $\rho = 2r\cos\theta$ by double integration.



*Solution.* Summing up all the elements in a sector (as $OB$), the limits are 0 and $2r\cos\theta$; and summing up all such sectors, the limits are 0 and $\frac{\pi}{2}$ for the semicircle $OXB$. Substituting in $(D)$,

$$\frac{A}{2} = \int_0^{\frac{\pi}{2}}\int_0^{2r\cos\theta}\rho d\rho d\theta = \frac{\pi r^2}{2}, \text{ or,}$$

$$A = \pi r^2. \quad \textit{Ans.}$$

EXAMPLES

1. In the above example find the area by integrating first with respect to $\theta$.

2. Find by double integration the entire area of the cardioid $\rho = a(1 - \cos\theta)$.

*Ans.* $\frac{3\pi a^2}{2}$.

3. Find by double integration the entire area of the lemniscate $\rho^2 = a^2\cos 2\theta$.

*Ans.* $a^2$.

4. Find by double integration the area of that part of the parabola $\rho = a\sec^2\frac{\theta}{2}$ intercepted between the curve and its latus rectum.

*Ans.* $2\int_0^{\frac{\pi}{2}}\int_0^{a\sec^2\frac{\theta}{2}}\rho d\rho d\theta = \frac{8a^2}{3}$.

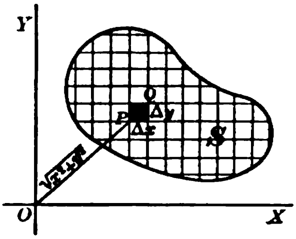
5. Find by double integration the area between the two circles $\rho = a\cos\theta$, $\rho = b\cos\theta$, $b > a$; integrating first with respect to $\rho$.

*Ans.* $\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{b\cos\theta}\rho d\rho d\theta = \frac{\pi}{8}(b^2 - a^2)$.

6. Solve the last problem by first integrating with respect to $\theta$.

*Ans.* $\int_a^b\int_0^{\text{arc}\cos\frac{\rho}{b}}\rho d\theta d\rho = \frac{\pi}{8}(b^2 - a^2)$.

**235. Moment of inertia. Rectangular coördinates.** Consider an element of the area of region $S$, as $PQ$. If the coördinates of $P$ are $(x, y)$, the distance of $P$ from $O$ is $\sqrt{x^2 + y^2}$. Multiplying the area of element, i.e.



$$\Delta x\Delta y,$$

by the square of the distance of $P$ from the origin, we have

$$(A) \qquad (x^2 + y^2)\Delta y\Delta x.$$

Then the value of

$$(B) \qquad \lim_{\substack{\Delta y = 0 \\ \Delta x = 0}} \sum\sum (x^2 + y^2)\Delta y \Delta x$$

*defines the moment of inertia of the area within the region S about the point O, when the summation is extended over the entire region.*

The product (*A*) is then an element of the moment of inertia.

Denoting this moment of inertia by *I*, we then have, as before,

$$(C) \qquad I = \iint\limits_S (x^2 + y^2)\, dy dx,^*$$

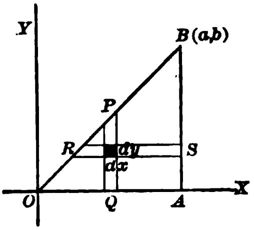
the limits of integration being determined in the same way as for finding the area (p. 403).

Ex. 1. Find $I$ over the area bounded by the lines $x = a$, $y = 0$, $y = \frac{b}{a}x$.

*Solution.* These lines bound a triangle $OAB$. Summing up all the elements in a vertical strip (as $PQ$), the $y$-limits are zero and $\frac{b}{a}x$ (found from the equation of the line $OB$). Summing up all such strips within the region (triangle), the $x$-limits are zero and $a (= OA)$. Hence

$$I = \int_0^a \int_0^{\frac{b}{a}x} (x^2 + y^2)\, dy dx$$

$$= ab\left(\frac{a^2}{4} + \frac{b^2}{12}\right). \quad \textit{Ans.}$$



If we suppose the triangle to be composed of horizontal strips (as $RS$),

$$I = \int_0^b \int_{\frac{ay}{b}}^a (x^2 + y^2)\, dx dy = ab\left(\frac{a^2}{4} + \frac{b^2}{12}\right). \quad \textit{Ans.}$$

## EXAMPLES

1. Find $I$ over the rectangle bounded by the lines $x = a$, $y = b$, and the coördinate axes. *Ans.* $\int_0^a \int_0^b (x^2 + y^2)\, dy dx = \frac{a^3b + ab^3}{3}$.

2. Find $I$ over the right triangle formed by the coördinate axes and the line joining the points $(a, 0)$, $(0, b)$. *Ans.* $\int_0^a \int_0^{\frac{b(a-x)}{a}} (x^2 + y^2)\, dy dx = \frac{ab(a^2 + b^2)}{12}$.

3. Find $I$ for the region within the circle $x^2 + y^2 = r^2$. *Ans.* $\frac{\pi r^4}{2}$.

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[*] From the result, p. 401, we may say that *the moment of inertia of the area within the region S is the value of the double integral of the function $f(x, y) = x^2 + y^2$ taken over that region.*

**4.** Find $I$ for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Compare result with preceding problem.

*Ans.* $\frac{\pi ab}{4}(a^2+b^2)$.

**5.** Find $I$ over the region between the straight line and a parabola with axis along $OX$, each of which joins the origin and the point $(a, b)$.

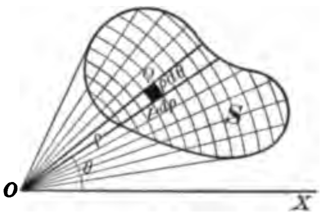
$$Ans.\ \int_0^a\int_{\frac{bx}{a}}^{b\sqrt{\frac{x}{a}}}(x^2+y^2)\,dy\,dx=\frac{ab}{4}\left(\frac{a^2}{7}+\frac{b^2}{5}\right).$$

**6.** Find $I$ over the region bounded by the parabola $y^2=4\,ax$, the line $x+y-3\,a=0$, and $OX$.

$$Ans.\ \int_0^a\int_0^{2\sqrt{ax}}(x^2+y^2)\,dy\,dx+\int_0^{3a}\int_0^{3a-x}(x^2+y^2)\,dy\,dx=\frac{314\,a^4}{35},$$

$$\text{or, }\int_0^{2a}\int_{\frac{y^2}{4a}}^{3a-y}(x^2+y^2)\,dx\,dy=\frac{314\,a^4}{35}.$$

**236. Moment of inertia. Polar coördinates.** Consider an element of the area of region $S$, as $PQ$. If the coördinates of $P$ are $(\rho, \theta)$, the distance of $P$ from $O$ is $\rho$. The area of the element in polar coördinates was found, on p. 406, to be



$$\rho\Delta\rho\Delta\theta+\tfrac{1}{2}\overline{\Delta\rho}^2\cdot\Delta\theta.$$

Multiplying this by the square of the distance of $P$ from the origin, we get

$$(A)\qquad \rho^2(\rho\Delta\rho\Delta\theta+\tfrac{1}{2}\overline{\Delta\rho}^2\cdot\Delta\theta).$$

Then, in conformity with the definition of moment of inertia $(=I)$ of the last section, we say that

$$(B)\qquad \lim_{\substack{\Delta\rho=0\\ \Delta\theta=0}}\sum\sum\rho^2(\rho\Delta\rho\cdot\Delta\theta+\tfrac{1}{2}\overline{\Delta\rho}^2\cdot\Delta\theta)$$

*defines the moment of inertia about O of the area within the region S.*

Or, passing to the limit (as under § 234, p. 406),

$$(C)\qquad I=\iint\limits_S\rho^3\,d\theta\,d\rho,$$

the limits of integration being determined in the same way as in finding the area (p. 407).

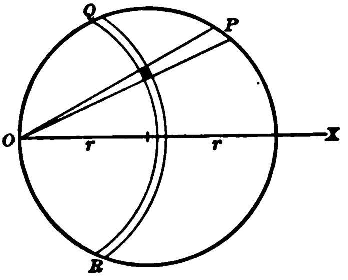
**Ex. 1.** Find $I$ over the region bounded by the circle $\rho = 2r\cos\theta$.

*Solution.* Summing up the elements in the triangular-shaped strip $OP$, the $\rho$-limits are zero and $2r\cos\theta$ (found from the equation of the circle). Summing up all such strips, the $\theta$-limits are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Hence

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2r\cos\theta} \rho^3 d\rho d\theta = \frac{3\pi r^4}{2}. \quad \textit{Ans.}$$

Summing up first the elements in a circular strip (as $QR$), we have

$$I = \int_0^{2r}\int_{-\arccos\frac{\rho}{2r}}^{\arccos\frac{\rho}{2r}} \rho^3 d\theta d\rho = \frac{3\pi r^4}{2}. \quad \textit{Ans.}$$



**EXAMPLES**

1. Find $I$ over the area bounded by the parabola $\rho = a\sec^2\frac{\theta}{2}$, its latus rectum, and the initial line $OX$.

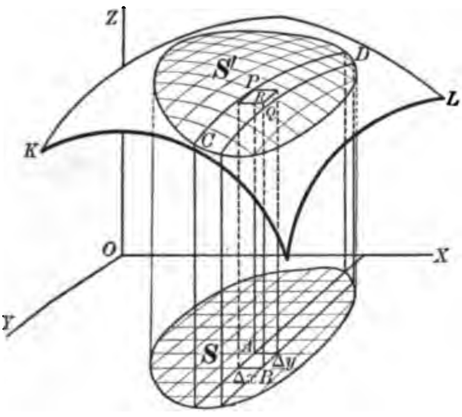
$$\textit{Ans.} \int_0^{\frac{\pi}{2}}\int_0^{a\sec^2\frac{\theta}{2}} \rho^3 d\rho d\theta = \frac{48a^4}{35}.$$

2. Find $I$ over the entire area of the cardioid $\rho = a(1-\cos\theta)$.

$$\textit{Ans.}\ 2\int_0^{\pi}\int_0^{a(1-\cos\theta)} \rho^3 d\rho d\theta = \frac{35\pi a^4}{16}.$$

3. Find $I$ over the area of the lemniscate $\rho^2 = a^2\cos 2\theta$. *Ans.* $\frac{\pi a^4}{8}$.

4. Find $I$ over the area bounded by one loop of the curve $\rho = a\cos 2\theta$.



### 237. General method for finding the areas of surfaces.

The method given in § 228 for finding the area of a surface applied only to surfaces of revolution. We shall now give a more general method. Let

$$(A) \qquad z = f(x, y)$$

be the equation of the surface $KL$ in figure, and suppose it is required to calculate the area of the region $S'$ lying on the surface.

Denote by $S$ the region on the $XOY$ plane, which is the orthogonal projection of $S'$ on that plane. Now pass planes parallel to $YOZ$ and $XOZ$ at common distances $\Delta x$ and $\Delta y$ respectively. As in § 231, these planes form truncated prisms (as $PB$) bounded at the top by a portion (as $PQ$) of the given surface whose projection on the $XOY$ plane is a rectangle of area $\Delta x \Delta y$ (as $AB$), which rectangle also forms the lower base of the prism, the coördinates of $P$ being $(x, y, z)$.

Now consider the plane tangent to the surface $KL$ at $P$. Evidently the same rectangle $AB$ is the projection on the $XOY$ plane of that portion of the tangent plane ($PR$) which is intercepted by the prism $PB$. Assuming $\gamma$ as the angle the tangent plane makes with the $XOY$ plane, we have

$$\text{area } AB = \text{area } PR \cdot \cos\gamma,$$

[The projection of a plane area upon a second plane is equal to the area of the portion projected multiplied by the cosine of the angle between the planes.]

or,

$$\Delta y \Delta x = \text{area } PR \cdot \cos\gamma.$$

But

$$\cos\gamma = \frac{1}{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}}};$$

[Cosine of angle between tangent plane, (70), p. 274, and $XOY$ plane found by method given in Solid Analytic Geometry.]

hence

$$\Delta y \Delta x = \frac{\text{area } PR}{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}}},$$

or,

$$\text{area } PR = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} \Delta y \Delta x,$$

which we take as the element of area of the region $S'$. We then define the area of the region $S'$ as

$$\underset{\substack{\Delta y = 0 \\ \Delta x = 0}}{\text{limit}} \sum\sum \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} \Delta y \Delta x,$$

the summation extending over the region $S$, as in § 231. Denoting by $A$ the area of the region $S'$, we have

$$(B) \qquad A = \int\int_S \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dy\, dx,$$

the limits of integration depending *on the projection on the XOY plane of the region whose area we wish to calculate.* Thus for (**B**) we chose our limits from the boundary curve or curves of the region $S$ in the $XOY$ plane precisely as we have been doing in the previous four sections.

If it is more convenient to project the required area on the $XOZ$ plane, use the formula

$$(C) \qquad A=\iint\limits_S\left[1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}}dz\,dx,$$

where the limits are found from the boundary of the region $S$, which is now the projection of the required area on the $XOZ$ plane.

Similarly we may use

$$(D) \qquad A=\iint\limits_S\left[1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2\right]^{\frac{1}{2}}dz\,dy,$$

the limits being found by projecting the required area on the $YOZ$ plane.

In some problems it is required to find the area of a portion of one surface intercepted by a second surface. In such cases the partial derivatives required for substitution in the formula should be found from the equation of the surface whose partial area is wanted.

Since the limits are found by projecting the required area on one of the coördinate planes, it should be remembered that —

*To find the projection of the area required on the XOY plane, eliminate z between the equations of the surfaces whose intersections form the boundary of the area.*

*Similarly we eliminate y to find the projection on the XOZ plane, and x to find it on the YOZ plane.*

This area of a surface gives a further illustration of *integration of a function over a given area.* Thus in (**B**), p. 412, we integrate the function

$$\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}}$$

over the projection on the $XOY$ plane of the required curvilinear surface.

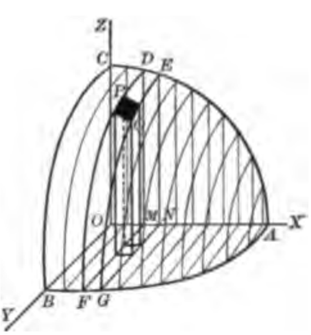
**Ex. 1.** Find the area of the surface of sphere $x^2 + y^2 + z^2 = r^2$ by double integration.

*Solution.* Let $ABC$ in the figure be one eighth of the surface of the sphere. Here

$$\frac{\partial z}{\partial x} = -\frac{x}{y}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z},$$

and
$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2 + y^2 + z^2}{z^2} = \frac{r^2}{r^2 - x^2 - y^2}.$$

The projection of the area required on the $XOY$ plane is $AOB$, a region bounded by $x = 0$, $(OB)$; $y = 0$, $(OA)$; $x^2 + y^2 = r^2$, $(BA)$.



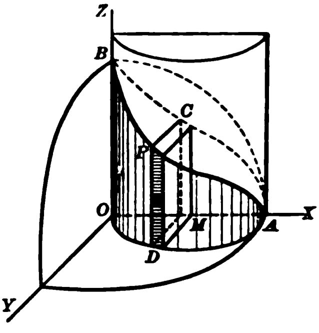
Integrating first with respect to $y$, we sum up all the elements along a strip (as $DEFG$) which is projected on the $XOY$ plane in a strip also (as $MNFG$); that is, our $y$-limits are zero and $MF (= \sqrt{r^2 - x^2})$. Then integrating with respect to $x$ sums up all such strips composing the surface $ABC$; that is, our $x$-limits are zero and $OA$ $(= r)$. Substituting in $(A)$,

$$\frac{A}{8} = \int_0^r \int_0^{\sqrt{r^2 - x^2}} \frac{r dy dx}{\sqrt{r^2 - x^2 - y^2}} = \frac{\pi r^2}{2},$$

or, $A = 4 \pi r^2$. *Ans.*

**Ex. 2.** The center of a sphere of radius $r$ is on the surface of a right cylinder, the radius of whose base is $\frac{r}{2}$. Find the surface of the cylinder intercepted by the sphere.

*Solution.* Taking the origin at the center of the sphere, an element of the cylinder for the $z$-axis, and a diameter of a right section of the cylinder for the $x$-axis, the equation of the sphere is $x^2 + y^2 + z^2 = r^2$, and of the cylinder $x^2 + y^2 = rx$. $ODAPB$ is evidently one fourth of the cylindrical surface required. Since this area projects into the semicircular arc $ODA$ on the $XOY$ plane, there is no region $S$ from which to determine our limits in this plane; hence we will project our area on, say, the $XOZ$ plane. Then the region $S$ over which we integrate is $OACB$, which is bounded by $z = 0$, $(OA)$; $x = 0$, $(OB)$; $z^2 + rx = r^2$, $(ACB)$; the last equation being found by eliminating $y$ between the equations of the two surfaces. Integrating first with respect to $z$ means that we sum up all the elements in a vertical strip (as $PD$), the $z$-limits being zero and $\sqrt{r^2 - rx}$. Then on integrating with respect to $x$ we sum up all such strips, the $x$-limits being zero and $r$.



Since the required surface lies on the cylinder, the partial derivatives required for formula $(C)$, p. 413, must be found from the equation of the cylinder.

Hence $$\frac{\partial y}{\partial x} = \frac{r - 2x}{2y},\ \frac{\partial y}{\partial z} = 0.$$

Substituting in $(C)$, p. 413,

$$\frac{A}{4} = \int_0^r \int_0^{\sqrt{r^2 - rx}} \left[1 + \left(\frac{r - 2x}{2y}\right)^2\right]^{\frac{1}{2}} dzdx.$$

Substituting the value of $y$ in terms of $x$ from the equation of the cylinder,

$$A = 2r\int_0^r \int_0^{\sqrt{r^2 - rx}} \frac{dzdx}{\sqrt{ax - x^2}} = 2r\int_0^r \frac{\sqrt{r^2 - rx}}{\sqrt{rx - x^2}} dx = 2r\int_0^r \sqrt{\frac{r}{x}} dx = 4a^2.$$

## EXAMPLES

1. In the preceding example find the surface of the sphere intercepted by the cylinder.

*Ans.* $4r\int_0^r \int_0^{\sqrt{rx - x^2}} \frac{dydx}{\sqrt{r^2 - x^2 - y^2}} = 2(\pi - 2)r^2.$

2. The axes of two equal right circular cylinders, $r$ being the radius of their bases, intersect at right angles; find the surface of one intercepted by the other.

*Hint.* Take $x^2 + z^2 = r^2$ and $x^2 + y^2 = r^2$ as equations of cylinders.

*Ans.* $8r\int_0^r \int_0^{\sqrt{r^2 - x^2}} \frac{dydx}{\sqrt{r^2 - x^2}} = 8r^2.$

3. Find the area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = 2ry$ lying within the paraboloid $y = ax^2 + bz^2$.

*Ans.* $\frac{2\pi r}{\sqrt{ab}}.$

4. Find the surface of the cylinder $x^2 + y^2 = r^2$ included between the plane $z = mx$ and the $XOY$ plane. *Ans.* $4r^2m.$

5. Find the surface of the cylinder $z^2 + (x \cos a + y \sin a)^2 = r^2$ which is situated in the positive compartment of coördinates.

*Hint.* The axis of this cylinder is the line $z = 0$, $x \cos a + y \sin a = 0$; and the radius of base is $r$.

*Ans.* $\frac{r^2}{\sin a \cos a}.$

6. The diameter of a sphere whose radius is $r$ is the axis of a right prism with square base of side $2a$. Find the surface of the sphere intercepted by the prism.

*Ans.* $8r\left(2a \arcsin\frac{a}{\sqrt{r^2 - a^2}} - r \arcsin\frac{a^2}{r^2 - a^2}\right).$

7. Find the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant intercepted between the planes $x = 0$, $y = 0$, $x = b$, $y = b$.

*Ans.* $a\left(2b \arcsin\frac{b}{\sqrt{a^2 - b^2}} - a \arcsin\frac{b^2}{a^2 - b^2}\right).$

**238. Volumes found by triple integration.** In many cases the volume of a solid bounded by surfaces whose equations are given may be calculated by means of three successive integrations, the process being merely an extension of the methods employed in the preceding sections of this chapter.

Suppose the solid in question be divided by planes parallel to the coördinate planes into rectangular parallelopipeds having the dimensions $\Delta z$, $\Delta y$, $\Delta x$. The volume of one of these parallelopipeds is

$$\Delta z \cdot \Delta y \cdot \Delta x,$$

and we choose it as the element of volume.

Now sum up all such elements within the region $R$ bounded by the given surfaces by first summing up all the elements in a column parallel to one of the coördinate axes; then sum up all such columns in a slice parallel to one of the coördinate planes containing that axis, and finally sum up all such slices within the region in question. The volume $V$ of the solid will then be the limit of this triple sum as $\Delta z$, $\Delta y$, $\Delta x$ each approaches zero as a limit. That is,

$$V = \underset{\substack{\Delta x = 0\\ \Delta y = 0\\ \Delta z = 0}}{\text{limit}} \underset{R}{\sum\sum\sum} \Delta z \Delta y \Delta x,$$

the summations being extended over the entire region $R$ bounded by the given surfaces. Or, what amounts to the same thing,

$$V = \iiint_R dz\, dy\, dx,$$

the limits of integration depending on the equations of the bounding surfaces.

Thus, by extension of the principle of § 232, p. 401, we speak of volume as the result of integrating the function $f(x, y, z) = 1$ *throughout a given region.* More generally many problems require the integration of a *variable* function of $x$, $y$, and $z$ throughout a given region, this being expressed by the notation

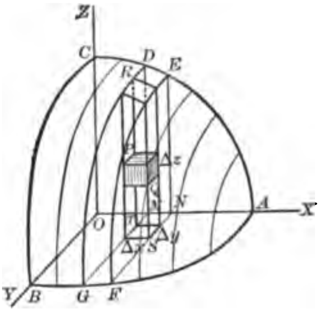
$$\iiint_R f(x, y, z)\, dz dy dx,$$

which is of course the limit of a triple sum analogous to the double sums we have already discussed. The method of evaluating this triple integral is precisely analogous to that already explained for double integrals in § 232, p. 401.

**Ex. 1.** Find the volume of that portion of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

which lies in the first octant.



*Solution.* Let $O-ABC$ be that portion of the ellipsoid whose volume is required, the equations of the bounding surfaces being

(1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $(ABC)$,

(2) $z=0$, $(OAB)$,

(3) $y=0$, $(OAC)$,

(4) $x=0$, $(OBC)$.

$PQ$ is an element, being one of the rectangular parallelopipeds with dimensions $\Delta z$, $\Delta y$, $\Delta x$ into which the planes parallel to the coördinate planes have divided the region.

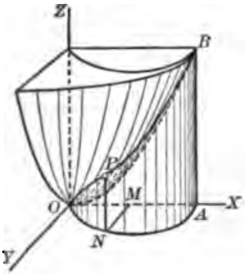
Integrating first with respect to $z$, we sum up all such elements in a column (as $RS$), the $z$-limits being zero [from (2)] and $TR=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ [from (1) by solving for $z$].

Integrating next with respect to $z$, we sum up all such columns in a slice (as $DEMNGF$), the $y$-limits being zero [from (3)] and $MG=b\sqrt{1-\frac{x^2}{a^2}}$ [from equation of the curve $AGB$, namely $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, by solving for $y$].

Lastly, integrating with respect to $x$, we sum up all such slices within the entire region $O-ABC$, the $x$-limits being zero [from (4)] and $OA=a$.

Hence
$$V=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dzdydx$$
$$=c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dydx$$
$$=\frac{\pi cb}{4a^2}\int_0^a(a^2-x^2)\,dx=\frac{\pi abc}{6}.$$

Therefore the volume of the entire ellipsoid is $\frac{4\pi abc}{3}$.



Ex. 2. Find the volume of the solid contained between the paraboloid of

revolution $x^2+y^2=az$,

the cylinder $x^2+y^2=2ax$,

and the plane $z=0$.

*Solution.* The $z$-limits are zero and $NP\left(=\frac{x^2+y^2}{a}\right.$, found by solving equation of paraboloid for $z$).

The $y$-limits are zero and $MN(=\sqrt{2ax-x^2}$, found by solving equation of cylinder for $y$).

The $x$-limits are zero and $OA$ $(=2a)$.

The above limits are for the solid $ONAB$, one half of the solid whose volume is required.

Hence $$\frac{V}{2}=\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\frac{x^2+y^2}{a}}dzdydx=\frac{3\pi a^3}{4}.$$

Therefore $$V=\frac{3\pi a^3}{4}$$

## EXAMPLES

1. Find the volume of the sphere $x^2+y^2+z^2=r^2$ by triple integration. *Ans.* $\frac{4\pi r^3}{3}$.

2. Find the volume of one of the wedges cut from the cylinder $x^2+y^2=r^2$ by the planes $z=0$ and $z=mx$. *Ans.* $2\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_0^{mx}dzdydx=\frac{2r^3m}{3}$.

3. Find the volume of a right elliptic cylinder whose axis coincides with the $x$-axis and whose altitude $=2a$, the equation of the base being $c^2y^2+b^2z^2=b^2c^2$.

*Ans.* $2\int_0^a\int_0^b\int_0^{\frac{c}{b}\sqrt{b^2-y^2}}dzdydx=2\pi abc$.

4. Find the entire volume bounded by the surface $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$. *Ans.* $\frac{abc}{90}$.

5. Find the entire volume bounded by the surface $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $\frac{4\pi a^3}{35}$.

6. Find the volume cut from a sphere of radius $a$ by a right circular cylinder with $b$ as radius of base, and whose axis passes through the center of the sphere.

*Ans.* $\frac{4\pi}{3}[a^3-(a^2-b^2)^{\frac{3}{2}}]$.

7. Find by triple integration the volume of the solid bounded by the planes $x=a$, $y=b$, $z=mx$ and the coördinate planes $XOY$ and $XOZ$. *Ans.* $\frac{1}{2}mba^2$.

8. The center of a sphere of radius $r$ is on the surface of a right circular cylinder the radius of whose basis is $\frac{r}{2}$. Find the volume of the portion of the cylinder intercepted by the sphere. *Ans.* $\frac{2}{3}(\pi-\frac{4}{3})r^3$.

9. Find the volume bounded by the hyperbolic paraboloid $cz=xy$, the $XOY$ plane, and the planes $x=a_1$, $x=a_2$, $y=b_1$, $y=b_2$. *Ans.* $\frac{(a_2^2-a_1^2)(b_2^2-b_1^2)}{4c}$.

10. Find the volume common to the two cylinders $x^2+y^2=r^2$ and $x^2+z^2=r^2$. *Ans.* $\frac{16r^3}{3}$.

**11.** Find the volume bounded by the plane $z = 0$, the cylinder

$$(x - a)^2 + (y - b)^2 = r^2,$$

and the hyperbolic paraboloid $xy = cz$. *Ans.* $\frac{\pi abr^2}{c}$.

**12.** Find the volume of the solid bounded by the surfaces

$$z = 0,\ x^2 + y^2 = 4\,az,\ x^2 + y^2 = 2\,cx.$$ *Ans.* $\frac{3\,\pi c^4}{8\,a}$.

**13.** Find the volume included by the surfaces $y^2 + z^2 = 4\,ax$ and $x - z = a$. *Ans.* $8\,\pi a^3$.

**14.** Find the volume of the solid in the first octant bounded by $xy = az$ and $x + y + z = a$. *Ans.* $\left(\frac{17}{12} - \log 4\right)a^3$.

**15.** Find the volume included between the plane $z = 0$, the cylinder $y^2 = 2\,cx - x^2$, and the paraboloid $ax^2 + by^2 = 2\,z$. *Ans.* $\frac{\pi c^4}{8}\left(\frac{5}{a^4} + \frac{1}{b}\right)$.

**16.** Find the entire volumes of the solids bounded by the following surfaces:

(a) $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1.$ *Ans.* $\frac{4\,\pi abc}{35}$.

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} = 1.$ *Ans.* $\frac{8\,\pi abc}{5}$.

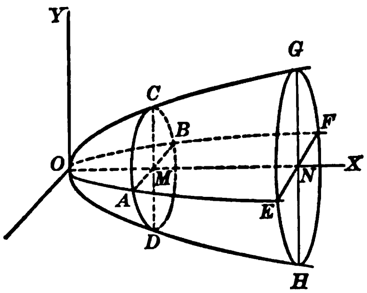
(c) $(x^2 + y^2 + z^2)^3 = 27\,a^3xyz.$ *Ans.* $\frac{9\,a^3}{2}$.

(d) $(x^2 + y^2 + z^2 + c^2 - a^2)^2 = 4\,c^2(x^2 + y^2).$ *Ans.* $2\,\pi^2ca^2$.

(e) $(x^2 + y^2 + z^2)^2 = cxyz.$ *Ans.* $\frac{c^3}{360}$.

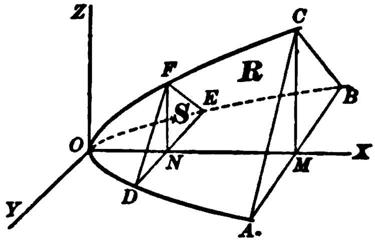
(f) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}.$ *Ans.* $\frac{\pi^2abc}{4\sqrt{2}}$.

**239. Miscellaneous applications of the Integral Calculus.** In § 227 it was shown how to calculate the volume of a solid of revolution by means of a single integration. Evidently we may consider a solid of revolution as generated by a moving circle of varying radius whose center lies on the axis of revolution and whose plane is perpendicular to it. Thus in the figure the circle *ACBD*, whose plane is perpendicular to *OX*, may be supposed to generate the solid of revolution $O - EGFH$, while its center moves from *O* to *N*, the radius *MC* $(= y)$ varying



continuously with $OM (= x)$ in a manner determined by the equation of the plane curve that is being revolved.

We will now show how this idea may be extended to the calculation of volumes that are not solids of revolution when it is possible to express the area of parallel plane sections of the solid as a function of their distances from a fixed point.



Suppose we choose sections of a solid perpendicular to $OX$ and take the origin as our fixed point. Assuming $FDE$ as such a section, we have from $(D)$, p. 404, the area

$$A = \int\int_S dz dy,$$

$x$ being regarded as constant and the limits of integration being extended over the area $S$, $(DFE)$. If the area of $DEF$ is expressible as a function of its distance from the origin $(= x)$, we then have

$$(A) \qquad \int\int_S dz dy = f(x).$$

But from § 238, p. 416, the volume of the entire solid is

$$V = \int\int\int_R dz dy dx = \int \left[ \int\int_S dz dy \right] dx.$$

Hence, substituting from $(A)$, we have

$$(B) \qquad V = \int f(x) dx,$$

where $f(x)$ is the area of a section of the solid perpendicular to $OX$ expressed in terms of its distance $(= x)$ from the origin, the $x$-limits being chosen so as to extend over the entire region $R$ occupied by the solid.

Evidently the solid $O-ABC$ may be considered as being generated by the continuously varying plane section $DEF$ as $ON (= x)$ varies from zero to $OM$. The following examples will further illustrate this principle.

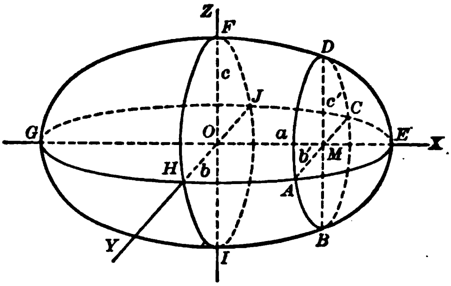
**Ex. 1.** Calculate the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

by means of a single integration.

*Solution.* Consider a section of the ellipsoid perpendicular to $OX$, as $ABCD$ with semiaxes $b'$ and $c'$. The equation of the ellipse $HEJG$ in the $XOY$ plane is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Solving this for $y\ (=b')$ in terms of $x\ (=OM)$ gives

$$b'=\frac{b}{a}\sqrt{a^2-x^2}.$$

Similarly from the equation of the ellipse $EFGI$ in the $XOZ$ plane we get

$$c'=\frac{c}{a}\sqrt{a^2-x^2}.$$

Hence the area of the ellipse (section) $ABCD$ is

$$\pi b'c'=\frac{\pi bc}{a^2}(a^2-x^2)=f(x).$$

Substituting in $(B)$, p. 420,

$$V=\frac{\pi bc}{a^2}\int_{-a}^{+a}(a^2-x^2)\,dx=\frac{4}{3}\pi abc.\quad \textit{Ans.}$$

We may then think of the ellipsoid as being generated by a variable ellipse $ABCD$ moving from $G$ to $E$, its center always on $OX$ and its plane perpendicular to $OX$.

**Ex. 2.** Find the volume of a right conoid with circular base, the radius of base being $r$ and altitude $a$.

*Solution.* Placing the conoid as shown in the figure, consider a section $PQR$ perpendicular to $OX$. This section is an isosceles triangle; and since

$$RM=\sqrt{2rx-x^2}$$

(found by solving $x^2+y^2=2rx$, the equation of the circle $ORAQ$, for $y$) and

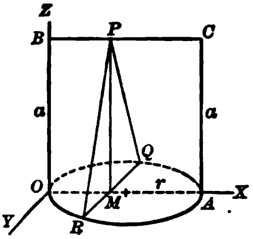
$$MP=a,$$

the area of the section is

$$a\sqrt{2rx-x^2}=f(x).$$

Substituting in $(B)$, p. 420,

$$V=a\int_0^{2r}\sqrt{2rx-x^2}\,dx=\frac{\pi r^2a}{2}.\quad \textit{Ans.}$$



This is one half the volume of the cylinder of the same base and altitude.

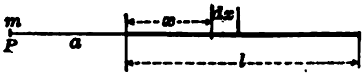
Ex. 3. A rectangle moves from a fixed point, one side varying as the distance from this point, and the other as the square of this distance. What is the volume generated while the rectangle moves a distance of 2 ft.? *Ans.* $4\frac{1}{2}$ cu. ft.

Ex. 4. On the double ordinates of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, isosceles triangles of vertical angle 90° are described in planes perpendicular to that of the ellipse. Find the volume of the solid generated by supposing such a variable triangle moving from one extremity to the other of the major axis of the ellipse. *Ans.* $\frac{4ab^2}{3}$.

Ex. 5. Given a right circular cylinder of altitude $a$ and radius of base $r$. Through a diameter of the upper base pass two planes which touch the lower base on opposite sides. Find the volume of the cylinder included between the two planes. *Ans.* $(\pi-\frac{4}{3})ar^2$.

Ex. 6. Two cylinders of equal altitude $a$ have a circle of radius $r$ for their common upper base. Their lower bases are tangent to each other. Find the volume common to the two cylinders. *Ans.* $\frac{4r^2a}{3}$.

Ex. 7. Determine the amount of attraction exerted by a thin, straight, homogeneous rod of uniform thickness, of length $l$, and of mass $M$, upon a material point $P$ of mass $m$ situated at a distance of $a$ from one end of the rod in its line of direction.



*Solution.* Suppose the rod to be divided into equal infinitesimal portions (elements) of length $dx$.

$$\frac{M}{l}=\text{mass of a unit length of rod};$$

hence

$$\frac{M}{l}dx=\text{mass of any element.}$$

Newton's Law for measuring the attraction between any two masses is

$$\text{force of attraction}=\frac{\text{product of masses}}{(\text{distance between them})^2};$$

therefore the force of attraction between the particle at $P$ and an element of the rod is

$$\frac{\frac{M}{l}mdx}{(x+a)^2},$$

which is then an *element of the force of attraction required.* The total attraction between the particle at $P$ and the rod being the sum of all such elements between $x=0$ and $x=l$, we have

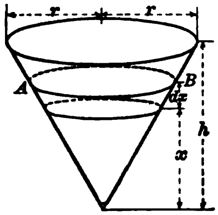
$$\text{force of attraction}=\int_0^l\frac{\frac{M}{l}mdx}{(x+a)^2}$$

$$=\frac{Mm}{l}\int_0^l\frac{dx}{(x+a)^2}=-\frac{Mm}{a(a+l)}. \quad \textit{Ans.}$$

**Ex.** 8. A vessel in the form of a right circular cone is filled with water. If $h$ is its height and $r$ the radius of base, what time will it require to empty itself through an orifice of area $a$ at the vertex?

*Solution.* Neglecting all hurtful resistances, it is known that the velocity of discharge through an orifice is that acquired by a body falling freely from a height equal to the depth of the water. If then $x$ denote depth of water,

$$v = \sqrt{2gx}.$$



Denote by $dQ$ the volume of water discharged in time $dt$, and by $dx$ the corresponding fall of surface. The volume of water discharged through the orifice in a unit of time is

$$a\sqrt{2gx},$$

being measured as a right cylinder of area of base $a$ and altitude $v\,(=\sqrt{2gx})$. Therefore in time $dt$,

$$(A)\qquad dQ = a\sqrt{2gx}\,dt.$$

Denoting by $S$ the area of surface of water when depth is $x$, we have, from Geometry,

$$\frac{S}{\pi r^2} = \frac{x^2}{h^2}, \text{ or, } S = \frac{\pi r^2 x^2}{h^2}.$$

But the volume of water discharged in time $dt$ may also be considered as the volume of cylinder $AB$ of area of base $S$ and altitude $dx$; hence

$$(B)\qquad dQ = S\,dx = \frac{\pi r^2 x^2 dx}{h^2}.$$

Equating $(A)$ and $(B)$ and solving for $dt$,

$$dt = \frac{\pi r^2 x^2 dx}{ah^2\sqrt{2gx}}.$$

Therefore

$$t = \int_0^h \frac{\pi r^2 x^2 dx}{ah^2\sqrt{2gx}} = \frac{2\pi r^2\sqrt{h}}{5a\sqrt{2g}}. \quad \textit{Ans.}$$

# CHAPTER XXXII

## ORDINARY DIFFERENTIAL EQUATIONS*

**240. Differential equations. Order and degree.** A differential equation is an equation involving derivatives or differentials. Differential equations have been frequently employed in this book, the following being examples.

(1) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0.$ Ex. 1, p. 155

(2) $\left(3a\dfrac{dy}{dx}+2\right)\left(\dfrac{d^2y}{dx^2}\right)^2=\left(a\dfrac{dy}{dx}+1\right)\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}.$ Ex. 4, p. 157

(3) $\tan\psi\dfrac{d\rho}{d\theta}=\rho.$ (*A*), p. 98

(4) $\dfrac{d^2y}{dx^2}=12(2x-1).$ Ex. 1, p. 113

(5) $dy=\dfrac{b^2x}{a^2y}dx.$ Ex. 2, p. 145

(6) $d\rho=-\dfrac{a^2\sin 2\theta}{\rho}d\theta.$ Ex. 3, p. 145

(7) $d^2y=(20x^3-12x)dx^2.$ Ex. 1, p. 146

(8) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=5u.$ Ex. 7, p. 197

(9) $x\dfrac{\partial^2u}{\partial x^2}+y\dfrac{\partial^2u}{\partial x\partial y}=2\dfrac{\partial u}{\partial x}.$ Ex. 8, p. 207

(10) $\dfrac{\partial^3u}{\partial x\partial y\partial z}=(1+3xyz+x^2y^2z^2)u.$ Ex. 7, p. 207

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* A few types only of differential equations are treated in this chapter, namely such as the student is likely to encounter in elementary work in Mechanics and Physics.

In fact, all of Chapter XIII in the Differential Calculus and all of Chapter XXV in the Integral Calculus treats of differential equations.

An *ordinary differential equation* involves only one independent variable. The first seven of the above examples are ordinary differential equations.

A *partial differential equation* involves more than one independent variable, as (8), (9), (10).

In this chapter we shall deal with ordinary differential equations only.

The *order* of a differential equation is that of the highest derivative (or differential) in it. Thus (3), (5), (6), (8) are of the *first order;* (1), (4), (7) are of the *second order;* and (2), (10) are of the *third order.*

The *degree* of a differential equation which is algebraic in the derivatives (or differentials) is the power of the highest derivative (or differential) in it when the equation is free from radicals and fractions. Thus all the above are examples of differential equations of the *first degree* except (2), which is of the *second degree.*

**241. Solutions of differential equations. Constants of integration.** A *solution* or *integral* of a differential equation is a relation between the variables involved by which the equation is identically satisfied. Thus

$$(A) \qquad y = c_1 \sin x$$

is a solution of the differential equation

$$(B) \qquad \frac{d^2y}{dx^2} + y = 0.$$

For, differentiating (*A*),

$$(C) \qquad \frac{d^2y}{dx^2} = - c_1 \sin x.$$

Now, if we substitute (*A*) and (*C*) in (*B*), we get

$$- c_1 \sin x + c_1 \sin x = 0,$$

showing that (*A*) satisfies (*B*) identically. Here $c_1$ is an arbitrary constant. In the same manner

$$(D) \qquad y = c_2 \cos x$$

may be shown to be a solution of $(B)$ for any value of $c_2$. The relation

$$(E) \qquad y = c_1 \sin x + c_2 \cos x$$

is a still more general solution of $(B)$. In fact, by giving particular values to $c_1$ and $c_2$ it is seen that the solution $(E)$ includes the solutions $(A)$ and $(D)$.

The arbitrary constants $c_1$ and $c_2$ appearing in these solutions are called *constants of integration.* A solution, such as $(E)$, which contains a number of arbitrary essential constants equal to the order of the equation (in this case two) is called the *general solution* or the *complete integral.** Solutions obtained therefrom by giving particular values to the constants are called *particular solutions* or *particular integrals.*

The solution of a differential equation is considered as having been effected when it has been reduced to an expression involving integrals, whether the actual integrations can be effected or not.

**242. Verification of the solutions of differential equations.** Before taking up the problem of solving differential equations it is best to further familiarize the student with what is meant by the *solution* of a differential equation by verifying a number of given solutions.

Ex. 1. Show that

$$(1) \qquad y = c_1 x \cos \log x + c_2 x \sin \log x + x \log x$$

is a solution of the differential equation

$$(2) \qquad x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x.$$

*Solution.* Differentiating (1), we get

$$(3) \qquad \frac{dy}{dx} = (c_2 - c_1) \sin \log x + (c_2 + c_1) \cos \log x + \log x + 1.$$

$$(4) \qquad \frac{d^2y}{dx^2} = -(c_2 + c_1) \frac{\sin \log x}{x} + (c_2 - c_1) \frac{\cos \log x}{x} + \frac{1}{x}.$$

Substituting (1), (3), (4) in (2), we find that the equation is identically satisfied.

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* It is shown in works on Differential Equations that the general solution has $n$ arbitrary constants when the differential equation is of the $n$th order.

### EXAMPLES

**Verify** the following solutions of the corresponding differential equations.

| | *Differential equations* | *Solutions* |
|---|---|---|
| 1. | $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} - x\frac{dy}{dx} + y = 0.$ | $y = cx + c - c^2.$ |
| 2. | $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.$ | $y = 2cx + c^2.$ |
| 3. | $xy\left[1 - \left(\frac{dy}{dx}\right)^2\right] = (x^2 - y^2 - a^2)\frac{dy}{dx}.$ | $y^2 - cx + \frac{a^2c}{1+c} = 0.$ |
| 4. | $\frac{dy}{dx}\left(\frac{dy}{dx} + y\right) = x(x + y).$ | $(2y - x^2 - c)[\log(x + y - 1) + x - c] = 0.$ |
| 5. | $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = e^x.$ | $y = (c_1 + c_2x)e^{kx} + \frac{e^x}{(k-1)^2}.$ |
| 6. | $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0.$ | $y = (c_1 + c_2x + c_3x^2 + c_4x^3)e^x.$ |
| 7. | $(x + y)^2\frac{dy}{dx} = a^2.$ | $y - a \arctan\frac{x+y}{a} = c.$ |
| 8. | $x\frac{dy}{dx} - y + x\sqrt{x^2 - y^2} = 0.$ | $\arcsin\frac{y}{x} = c - x.$ |
| 9. | $\frac{dy}{dx} + y\cos x = \frac{\sin 2x}{2}.$ | $y = \sin x - 1 + ce^{-\sin x}.$ |
| 10. | $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$ | $y = c_1e^{a \arcsin x} + c_2e^{-a \arcsin x}.$ |

**243. Differential equations of the first order and of the first degree.** Such an equation may be brought into the form $Mdx + Ndy = 0$, in which $M$ and $N$ are functions of $x$ and $y$. Differential equations coming under this head may be divided into the following types.

**Type I. Variables separable.** When the terms of a differential equation can be so arranged that it takes on the form

$$(A) \qquad f(x)\,dx + F(y)\,dy = 0,$$

where $f(x)$ is a function of $x$ alone and $F(y)$ is a function of $y$ alone, the process is called *separation of the variables*, and the solution is obtained by direct integration. Thus, integrating $(A)$, we get the general solution

$$(B) \qquad \int f(x)\,dx + \int F(y)\,dy = c,$$

where $c$ is an arbitrary constant.

Equations which are not given in the simple form $(A)$ may often be brought into that form by means of the following **rule for separating the variables.**

**First step.** *Clear of fractions, and if the equation involves derivatives, multiply through by the differential of the independent variable.*

**Second step.** *Collect all the terms containing the same differential into a single term. If then the equation takes on the form*

$$XYdx + X'Y'dy = 0,$$

*where $X$, $X'$ are functions of $x$ alone, and $Y$, $Y'$ are functions of $y$ alone, it may be brought to the form $(A)$ by dividing through by $X'Y$.*

**Third step.** *Integrate each part separately, as in $(B)$.*

**Ex. 1.** Solve the equation

$$\frac{dy}{dx} = \frac{1+y^2}{(1+x^2)xy}.$$

*Solution.*

*First step.* $(1+x^2)xydy = (1+y^2)dx.$

*Second step.* $(1+y^2)dx - x(1+x^2)ydy = 0.$

To separate the variables we now divide by $x(1+x^2)(1+y^2)$, giving

$$\frac{dx}{x(1+x^2)} - \frac{ydy}{1+y^2} = 0.$$

*Third step.* $\displaystyle\int \frac{dx}{x(1+x^2)} - \int \frac{ydy}{1+y^2} = C,$

$$\int \frac{dx}{x} - \int \frac{xdx}{1+x^2} - \int \frac{ydy}{1+y^2} = C, \qquad p.\ 323$$

$$\log x - \tfrac{1}{2}\log(1+x^2) - \tfrac{1}{2}\log(1+y^2) = C,$$

$$\log(1+x^2)(1+y^2) = 2\log x - 2C.$$

This result may be written in more compact form if we replace $-2C$ by $\log c$, i.e. we simply give a new form to the arbitrary constant. Our solution then becomes

$$\log(1+x^2)(1+y^2) = \log x^2 + \log c,$$

$$\log(1+x^2)(1+y^2) = \log cx^2,$$

$$(1+x^2)(1+y^2) = cx^2. \quad \textit{Ans.}$$

**Ex. 2.** Solve the equation

$$a\left(x\frac{dy}{dx} + 2y\right) = xy\frac{dy}{dx}.$$

*Solution.*

*First step.* $axdy + 2aydx = xydy.$

*Second step.* $2aydx + x(a-y)dy = 0.$

To separate the variables we divide by $xy$,

$$\frac{2adx}{x} + \frac{(a-y)dy}{y} = 0.$$

*Third step.*

$$2a\int\frac{dx}{x}+a\int\frac{dy}{y}-\int dy=C,$$

$$2a\log x+a\log y-y=C,$$

$$a\log x^2y=C+y,$$

$$\log_e x^2y=\frac{C}{a}+\frac{y}{a}.$$

By passing from logarithms to exponentials this result may be written in the form

$$x^2y=e^{\frac{C}{a}+\frac{y}{a}},$$

or,

$$x^2y=e^{\frac{C}{a}}\cdot e^{\frac{y}{a}}.$$

Denoting the constant $e^{\frac{C}{a}}$ by $c$, we get our solution in the form.

$$x^2y=ce^{\frac{y}{a}}.\quad \textit{Ans.}$$

## EXAMPLES

| *Differential equations* | *Solutions* |
|---|---|
| 1. $ydx-xdy=0.$ | $y=cx.$ |
| 2. $(1+y)\,dx-(1-x)\,dy=0.$ | $(1+y)(1-x)=c.$ |
| 3. $(1+x)\,ydx+(1-y)\,xdy=0.$ | $\log xy+x-y=c.$ |
| 4. $(x^2-a^2)\,dy-ydx=0.$ | $y^{2a}=c\dfrac{x-a}{x+a}.$ |
| 5. $(x^2-yx^2)\dfrac{dy}{dx}+y^2+xy^2=0.$ | $\dfrac{x+y}{xy}+\log\dfrac{y}{x}=c.$ |
| 6. $u^2dv+(v-a)\,du=0.$ | $v-a=ce^{\frac{1}{u}}.$ |
| 7. $\dfrac{du}{dv}=\dfrac{1+u^2}{1+v^2}.$ | $u=\dfrac{v+c}{1-cv}.$ |
| 8. $(1+s^2)\,dt-t^{\frac{1}{2}}ds=0.$ | $2\,t^{\frac{1}{2}}-\text{arc tan}\, s=c.$ |
| 9. $d\rho+\rho\tan\theta d\theta=0.$ | $\rho=c\cos\theta.$ |
| 10. $\sin\theta\cos\phi d\theta-\cos\theta\sin\phi d\phi=0.$ | $\cos\phi=c\cos\theta.$ |
| 11. $\sec^2\theta\tan\phi d\theta+\sec^2\phi\tan\theta d\phi=0.$ | $\tan\theta\tan\phi=c.$ |
| 12. $\sec^2\theta\tan\phi d\phi+\sec^2\phi\tan\theta d\theta=0.$ | $\sin^2\theta+\sin^2\phi=c.$ |
| 13. $xydx-(a+x)(b+y)\,dy=0.$ | $x-y=c+\log(a+x)^a y^b.$ |
| 14. $(1+x^2)\,dy-\sqrt{1-y^2}\,dx=0.$ | $\text{arc sin}\, y-\text{arc tan}\, x=c.$ |
| 15. $\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0.$ | $y\sqrt{1-x^2}+x\sqrt{1-y^2}=c.$ |
| 16. $3\,e^x\tan ydx+(1-e^x)\sec^2 ydy.$ | $\tan y=c\,(1-e^x)^3.$ |

**Type II. Homogeneous equations.** The differential equation

$$Mdx + Ndy = 0$$

is said to be homogeneous when $M$ and $N$ are homogeneous functions of $x$ and $y$ of the same degree.* Such differential equations may be solved by making the substitution

$$y = vx.$$

This will give a differential equation in $v$ and $x$ in which the variables are separable, and hence we may follow the rule on p. 428.

Ex. 1. Solve the equation

$$y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}.$$

*Solution.*

$$y^2dx + (x^2 - xy)\,dy = 0.$$

Since this is a homogeneous differential equation we transform it by means of the substitution $y = vx$. Hence $dy = vdx + xdv$,

and our equation becomes

$$v^2x^2dx + (x^2 - vx^2)(vdx + xdv) = 0,$$

$$x^2vdx + x^3(1 - v)\,dv = 0.$$

To separate the variables divide by $vx^3$.

This gives

$$\frac{dx}{x} + \frac{(1 - v)\,dv}{v} = 0,$$

$$\int\frac{dx}{x} + \int\frac{dv}{v} - \int dv = C,$$

$$\log x + \log v - v = C,$$

$$\log_e vx = C + v,$$

$$vx = e^{C+v} = e^C \cdot e^v,$$

$$vx = ce^v.$$

But $v = \frac{y}{x}\cdot$ Hence the solution is

$$y = ce^{\frac{y}{x}}. \quad \textit{Ans.}$$

## EXAMPLES

| *Differential equations* | *Solutions* |
|---|---|
| 1. $(x + y)\,dx + xdy = 0.$ | $x^2 + 2\,xy = c.$ |
| 2. $(x + y)\,dx + (y - x)\,dy = 0.$ | $\log(x^2 + y^2)^{\frac{1}{2}} - \text{arc tan}\,\frac{y}{x} = c.$ |
| 3. $xdy - ydx = \sqrt{x^2 + y^2}\,dx.$ | $1 + 2\,cy - c^2x^2 = 0.$ |
| 4. $(8\,y + 10\,x)\,dx + (5\,y + 7\,x)\,dy = 0.$ | $(x + y)^2(2\,x + y)^3 = c.$ |

---

* A function of $x$ and $y$ is said to be *homogeneous* in the variables if the result of replacing $x$ and $y$ by $\lambda x$ and $\lambda y$ ($\lambda$ being arbitrary) reduces to the original function multiplied by some power of $\lambda$. This power of $\lambda$ is called the *degree* of the original function.

| | *Differential equations* | *Solutions* |
|---|---|---|
| 5. | $(2\sqrt{st} - s)\,dt + t\,ds = 0.$ | $te^{\sqrt{\frac{s}{t}}} = c.$ |
| 6. | $(t - s)\,dt + t\,ds = 0.$ | $te^{\frac{s}{t}} = c.$ |
| 7. | $x\cos\frac{y}{x}\cdot\frac{dy}{dx} = y\cos\frac{y}{x} - x.$ | $xe^{\sin\frac{y}{x}} = c.$ |
| 8. | $x\cos\frac{y}{x}(ydx + xdy) = y\sin\frac{y}{x}(xdy - ydx).$ | $xy\cos\frac{y}{x} = c.$ |

**Type III. Linear equations.** A differential equation is said to be *linear* if the equation is of the first degree in the dependent variable (usually $y$) and its derivatives (or differentials). The linear differential equation of the first order is of the form

$$(A) \qquad \frac{dy}{dx} + Py = Q,$$

where $P$, $Q$ are functions of $x$ alone, or constants.

To integrate $(A)$, let

$$(B) \qquad y = uz,$$

where $z$ is a new variable and $u$ is a function of $x$ to be determined. Differentiating $(B)$,

$$(C) \qquad \frac{dy}{dx} = u\frac{dz}{dx} + z\frac{du}{dx}. \qquad \text{By V, p. 46}$$

Substituting $(C)$ and $(B)$ in $(A)$, we get

$$u\frac{dz}{dx} + z\frac{du}{dx} + Puz = Q, \text{ or,}$$

$$(D) \qquad u\frac{dz}{dx} + \left(\frac{du}{dx} + Pu\right)z = Q.$$

Now let us determine, if possible, the function $u$ such that the term in $z$ shall drop out. This means that the coefficient of $z$ must vanish, that is,

$$\frac{du}{dx} + Pu = 0.$$

Then

$$\frac{du}{u} = -Pdx,$$

and

$$\log_e u = -\int Pdx, \text{ giving}$$

$$(E) \qquad u = c_1 e^{-\int Pdx}.$$

Equation $(D)$ then becomes

$$u\frac{dz}{dx} = Q.$$

To find $z$ from the last equation, substitute in it the value of $u$ from $(E)$ and integrate. This gives

$$c_1 e^{-\int P dx}\frac{dz}{dx} = Q,$$

$$c_1 dz = Qe^{\int P dx}dx,$$

$$(F) \qquad c_1 z = \int Qe^{\int P dx}dx + C.$$

The solution of $(A)$ is then found by substituting the values of $u$ and $z$ from $(E)$ and $(F)$ in $(B)$. This gives

$$(G) \qquad y = e^{-\int P dx}\left(\int Qe^{\int P dx}dx + C\right).$$

The proof of the correctness of $(G)$ is immediately established by substitution in $(A)$. In solving examples coming under this head the student is advised to find the solution by following the method illustrated above, rather than by using $(G)$ as a formula.

**Ex. 1.** Solve the equation

$$(1) \qquad \frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^{\frac{5}{2}}.$$

*Solution.* This is evidently in the linear form $(A)$, where

$$P = -\frac{2}{x+1} \text{ and } Q = (x+1)^{\frac{5}{2}}.$$

Let $y = uz$; then $\frac{dy}{dx} = u\frac{dz}{dx} + z\frac{du}{dx}$. Substituting in the given equation (1), we get

$$u\frac{dz}{dx} + z\frac{du}{dx} - \frac{2uz}{1+x} = (x+1)^{\frac{5}{2}}, \text{ or,}$$

$$(2) \qquad u\frac{dz}{dx} + \left(\frac{du}{dx} - \frac{2u}{1+x}\right)z = (x+1)^{\frac{5}{2}}.$$

Now to determine $u$ we place the coefficient of $z$ equal to zero. This gives

$$\frac{du}{dx} - \frac{2u}{1+x} = 0,$$

$$\frac{du}{u} = \frac{2dx}{1+x},$$

$$\log_e u = 2\log(1+x),$$

$$(3) \qquad u = e^{\log(1+x)^2} = (1+x)^2.^*$$

---

* Since $\log_e u = \log_e e^{\log(1+x)^2} = \log(1+x)^2 \cdot \log_e e = \log(1+x)^2$, it follows that $u = (1+x)^2$.

Equation (2) now becomes, since the term in $z$ drops out,

$$u\frac{dz}{dx} = (x + 1)^{\frac{5}{2}}.$$

$u$ by its value from (3),

$$\frac{dz}{dx} = (x + 1)^{\frac{1}{2}},$$

$$dz = (x + 1)^{\frac{1}{2}}dx,$$

$$z = \frac{2(x + 1)^{\frac{3}{2}}}{3} + C.$$

ing (4) and (3) in $y = uz$, we get the solution

$$y = \frac{2(x + 1)^{\frac{7}{2}}}{3} + C(x + 1)^2. \quad \textit{Ans.}$$

### EXAMPLES

| *rential equations* | *Solutions* |
|---|---|
| $\frac{2y}{x+1} = (x + 1)^3.$ | $2y = (x + 1)^4 + c(x + 1)^2.$ |
| $\cdot\frac{ay}{x} = \frac{x+1}{x}.$ | $y = cx^a + \frac{x}{1-a} - \frac{1}{a}.$ |
| $- x^2)dy + (2x^2 - 1)ydx = ax^3dx.$ | $y = ax + cx\sqrt{1 - x^2}.$ |
| 4. $dy - \frac{xydx}{1+x^2} = \frac{adx}{1+x^2}.$ | $y = ax + c(1 + x^2)^{\frac{1}{2}}.$ |
| 5. $\frac{ds}{dt}\cos t + s\sin t = 1.$ | $s = \sin t + c\cos t.$ |
| 6. $\frac{ds}{dt} + s\cos t = \frac{1}{2}\sin 2t.$ | $s = \sin t - 1 + ce^{-\sin t}.$ |

**Type IV. Equations reducible to the linear form.** Some equations that are not linear can be reduced to the linear form by means of a suitable transformation. One type of such equations is

$$(A) \qquad \frac{dy}{dx} + Py = Qy^n,$$

where $P$, $Q$ are functions of $x$ alone, or constants. Equation $(A)$ may be reduced to the linear form in $y$ and $z$ by means of the substitution $z = y^{-n+1}$. Such a reduction, however, is not necessary if we employ the same method for finding the solution as that given under Type III, p. 431. Let us illustrate this by means of an example.

**Ex. 1.** Solve the equation

$$\frac{dy}{dx} + \frac{y}{x} = a \log x \cdot y^2. \tag{1}$$

*Solution.* This is evidently in the form ($A$), where

$$P = \frac{1}{x},\quad Q = a \log x,\quad n = 2.$$

Let $y = uz$; then
$$\frac{dy}{dx} = u\frac{dz}{dx} + z\frac{du}{dx}.$$

Substituting in (1), we get

$$u\frac{dz}{dx} + z\frac{du}{dx} + \frac{uz}{x} = a \log x \cdot u^2z^2,$$

$$u\frac{dz}{dx} + \left(\frac{du}{dx} + \frac{u}{x}\right)z = a \log x \cdot u^2z^2. \tag{2}$$

Now to determine $u$ we place the coefficient of $z$ equal to zero. This gives

$$\frac{du}{dx} + \frac{u}{x} = 0,$$

$$\frac{du}{u} = -\frac{dx}{x},$$

$$\log u = -\log x = \log\frac{1}{x},$$

$$u = \frac{1}{x}. \tag{3}$$

Since the term in $z$ drops out, equation (2) now becomes

$$u\frac{dz}{dx} = a \log x \cdot u^2z^2,$$

$$\frac{dz}{dx} = a \log x \cdot uz^2.$$

Replacing $u$ by its value from (3),

$$\frac{dz}{dx} = a \log x \cdot \frac{z^2}{x},$$

$$\frac{dz}{z^2} = a \log x \cdot \frac{dx}{x},$$

$$-\frac{1}{z} = \frac{a(\log x)^2}{2} + C,$$

$$z = -\frac{2}{a(\log x)^2 + 2C}. \tag{4}$$

Substituting (4) and (3) in $y = uz$, we get the solution

$$y = -\frac{1}{x}\cdot\frac{2}{a(\log x)^2 + 2C},$$

or,
$$xy\left[a(\log x)^2 + 2C\right] + 2 = 0. \quad \textit{Ans.}$$

### EXAMPLES

| *Differential equations* | *Solutions* |
|---|---|
| 1. $\frac{dy}{dx} + xy = x^3y^3.$ | $y^{-2} = x^2 + 1 + ce^{x^2}.$ |
| 2. $(1 - x^2)\frac{dy}{dx} - xy = axy^2.$ | $y = (c\sqrt{1 - x^2} - a)^{-1}.$ |
| 3. $3y^2\frac{dy}{dx} - ay^3 = x + 1.$ | $y^3 = ce^{ax} - \frac{x+1}{a} - \frac{1}{a^2}.$ |
| 4. $\frac{dy}{dx}(x^2y^3 + xy) = 1.$ | $x[(2 - y^2)e^{\frac{y^2}{2}} + c] = e^{\frac{y^2}{2}}.$ |
| 5. $(y \log x - 1)y\,dx = x\,dy.$ | $y = (cx + \log x + 1)^{-1}.$ |
| 6. $y - \cos x\frac{dy}{dx} = y^2 \cos x(1 - \sin x).$ | $y = \frac{\tan x + \sec x}{\sin x + c}.$ |

**244. Differential equations of the $n$th order and of the first degree.** Under this head we will consider four types which are of importance in elementary work. They are special cases of *linear differential equations*, which we defined on p. 431.

**Type I. The linear differential equation**

$$(A) \qquad \frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_ny = 0,$$

*in which the coefficients $p_1, p_2, \cdots, p_n$ are constants.*

The substitution of $e^{rx}$ for $y$ in the first member gives

$$(r^n + p_1r^{n-1} + p_2r^{n-2} + \cdots + p_n)e^{rx}.$$

This expression vanishes for all values of $r$ which satisfy the equation

$$(B) \qquad r^n + p_1r^{n-1} + p_2r^{n-2} + \cdots + p_n = 0;$$

and therefore for each of these values of $r$, $e^{rx}$ is a solution of $(A)$. Equation $(B)$ is called the *auxiliary equation* of $(A)$. We observe that the coefficients are the same in both, the exponents in $(B)$ corresponding to the order of the derivatives in $(A)$, and $y$ in $(A)$ being replaced by 1. Let the roots of the auxiliary equation $(B)$ be $r_1, r_2, \cdots, r_n$; then

$$(C) \qquad e^{r_1x}, e^{r_2x}, \cdots, e^{r_nx}$$

are solutions of $(A)$. Moreover, if each one of the solutions be multiplied by an arbitrary constant, the products

$$(D) \qquad c_1e^{r_1x},\ c_2e^{r_2x},\ \cdots,\ c_ne^{r_nx}$$

are also found to be solutions.* And the sum of the solutions $(D)$, namely

$$(E) \qquad y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx},$$

may, by substitution, be shown to be a solution of $(A)$. Solution $(E)$ contains $n$ arbitrary constants and is the *general solution* the roots are all different), while $(C)$ are *particular solutions.*

CASE I. *When the auxiliary equation has imaginary roots.* Since imaginary roots occur in pairs, let one pair of such roots be

$$r_1 = a + bi, \quad r_2 = a - bi. \qquad i = \sqrt{-1}$$

The corresponding solution is

$$\begin{aligned} y &= c_1e^{(a+bi)x} + c_2e^{(a-bi)x} \\ &= e^{ax}(c_1e^{ibx} + c_2e^{-ibx}) \\ &= e^{ax}\{c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)\}\dagger \\ &= e^{ax}\{(c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx\}, \end{aligned}$$

or, $\quad y = e^{ax}(A\cos bx + B\sin bx),$

where $A$ and $B$ are arbitrary constants.

---

* Substituting $c_1e^{r_1x}$ for $y$ in $(A)$, the left-hand member becomes

$$(r_1{}^n + p_1r_1{}^{n-1} + p_2r_1{}^{n-2} + \cdots + p_n)c_1e^{r_1x}.$$

But this vanishes since $r_1$ is a root of $(B)$; hence $c_1e^{r_1x}$ is a solution of $(A)$. Similarly for the other roots.

† Replacing $x$ by $ibx$ in Ex. 1, p. 235, gives

$$e^{ibx} = 1 + ibx - \frac{b^2x^2}{\lfloor 2} - \frac{ib^3x^3}{\lfloor 3} + \frac{b^4x^4}{\lfloor 4} + \frac{ib^5x^5}{\lfloor 5} - \cdots, \text{ or,}$$

$$(1) \qquad e^{ibx} = 1 - \frac{b^2x^2}{\lfloor 2} + \frac{b^4x^4}{\lfloor 4} - \cdots + i\left(bx - \frac{b^3x^3}{\lfloor 3} + \frac{b^5x^5}{\lfloor 5} - \cdots\right);$$

and replacing $x$ by $-ibx$ gives

$$e^{-ibx} = 1 - ibx - \frac{b^2x^2}{\lfloor 2} + \frac{ib^3x^3}{\lfloor 3} + \frac{b^4x^4}{\lfloor 4} - \frac{ib^5x^5}{\lfloor 5} - \cdots, \text{ or,}$$

$$(2) \qquad e^{-ibx} = 1 - \frac{b^2x^2}{\lfloor 2} + \frac{b^4x^4}{\lfloor 4} - \cdots - i\left(bx - \frac{b^3x^3}{\lfloor 3} + \frac{b^5x^5}{\lfloor 5} - \cdots\right).$$

But, replacing $x$ by $bx$ in $(A)$, $(B)$, pp. 235, 236, we get

$$(3) \qquad \cos bx = 1 - \frac{b^2x^2}{\lfloor 2} + \frac{b^4x^4}{\lfloor 4} - \cdots.$$

$$(4) \qquad \sin bx = bx - \frac{b^3x^3}{\lfloor 3} + \frac{b^5x^5}{\lfloor 5} - \cdots.$$

Hence (1) and (2) become

$$e^{ibx} = \cos bx + i\sin bx, \quad e^{-ibx} = \cos bx - i\sin bx.$$

**CASE** II. *When the auxiliary equation has multiple roots.* Consider the linear differential equation of the third order

$$(F) \qquad \frac{d^3y}{dx^3}+p_1\frac{d^2y}{dx^2}+p_2\frac{dy}{dx}+p_3y=0,$$

where $p_1, p_2, p_3$ are constants. The corresponding auxiliary equation is

$$(G) \qquad r^3+p_1r^2+p_2r+p_3=0.$$

If $r_1$ is a root of $(G)$, we have shown that $e^{r_1x}$ is a solution of $(F)$. We will now show that if $r_1$ is a double root of $(G)$, then $xe^{r_1x}$ is also a solution of $(F)$. Replacing $y$ in the left-hand member of $(F)$ by $xe^{r_1x}$, we get

$$(H) \quad xe^{r_1x}(r_1^3+p_1r_1^2+p_2r_1+p_3)+e^{r_1x}(3r_1^2+2p_1r_1+p_2).$$

But since $r_1$ is a double root of $(G)$,

$$r_1^3+p_1r_1^2+p_2r_1+p_3=0,$$

and $$3r_1^2+2p_1r_1+p_2=0.$$ By § 82, p. 101

Hence $(H)$ vanishes, and $xe^{r_1x}$ is a solution of $(F)$. Corresponding to the double root $r$ we then have the two solutions

$$c_1e^{r_1x}, \quad c_2xe^{r_1x}.$$

More generally, if $r_1$ is a multiple root of the auxiliary equation $(B)$, p. 435, occurring $s$ times, then we may at once write down $s$ distinct solutions of the differential equation $(A)$, p. 435, namely

$$c_1e^{r_1x}, \ c_2xe^{r_1x}, \ c_3x^2e^{r_1x}, \ \cdots, \ c_sx^{s-1}e^{r_1x}.$$

In case $a+bi$ and $a-bi$ are each multiple roots of the auxiliary equation, occurring $s$ times, it follows that we may write down $2s$ distinct solutions of the differential equation, namely

$$c_1e^{ax}\cos bx, \ c_2xe^{ax}\cos bx, \ c_3x^2e^{ax}\cos bx, \ \cdots, \ c_sx^{s-1}e^{ax}\cos bx;$$

$$c_1'e^{ax}\sin bx, \ c_2'xe^{ax}\cos bx, \ c_3'x^2e^{ax}\cos bx, \ \cdots, \ c_s'x^{s-1}e^{ax}\cos bx.$$

Our results may now be summed up in the following **rule for solving differential equations of the type**

$$\frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+p_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+p_ny=0,$$

*where $p_1, p_2, \cdots, p_n$ are constants.*

**First step.** *Write down the corresponding auxiliary equation*

$$r^n + p_1 r^{n-1} + p_2 r^{n-2} + \cdots + p_n = 0.$$

**Second step.** *Solve completely the auxiliary equation.*

**Third step.** *From the roots of the auxiliary equation write down the corresponding particular solutions of the differential equation as follows:*

| AUXILIARY EQUATION | | DIFFERENTIAL EQUATION |
|---|---|---|
| (*a*) *Each distinct real root* $r_1$ | *gives* | *a particular solution* $e^{r_1 x}$. |
| (*b*) *Each distinct pair of imaginary roots* $a \pm bi$ | *gives* | *two particular solutions* $e^{ax}\cos bx$, $e^{ax}\sin bx$. |
| (*c*) *A multiple root occurring s times* | *gives* | *s particular solutions obtained by multiplying the particular solutions* (*a*) *or* (*b*) *by* $1, x, x^2, \cdots, x^{s-1}$. |

**Fourth step.** *Multiply each of the n* independent solutions by an arbitrary constant and add the results. This gives the complete solution.*

Ex. 1. Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = 0.$

*Solution.* Follow above rule.

*First step.* $r^3 - 3r^2 + 4 = 0$, auxiliary equation.

*Second step.* Solving, the roots are $-1, 2, 2$.

*Third step.* (a) The root $-1$ gives the solution $e^{-x}$.

(b) The double root 2 gives the two solutions $e^{2x}$, $xe^{2x}$.

*Fourth step.* General solution is

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x}. \quad \textit{Ans.}$$

Ex. 2. Solve $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 10\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 5y = 0.$

*Solution.* Follow above rule.

*First step.* $r^4 - 4r^3 + 10r^2 - 12r + 5 = 0$, auxiliary equation.

*Second step.* Solving, the roots are $1, 1, 1 \pm 2i$.

---

* A check on the accuracy of the work is found in the fact that the first three steps must give $n$ independent solutions.

**Third** *step*. (b) The pair of imaginary roots $1 \pm 2i$ gives the two solutions $e^x \cos 2x$, $e^x \sin 2x\,(a = 1,\ b = 2)$.

(c) The double root 1 gives the two solutions $e^x$, $xe^x$.

**Fourth** *step*. General solution is

$$y = c_1e^x + c_2xe^x + c_3e^x \cos 2x + c_4e^x \sin 2x,$$

or,

$$y = (c_1 + c_2x + c_3 \cos 2x + c_4 \sin 2x)\,e^x. \quad \textit{Ans.}$$

## EXAMPLES

| | *Differential equations* | *General solutions* |
|---|---|---|
| 1. | $\dfrac{d^2y}{dx^2} = 9y.$ | $y = c_1e^{3x} + c_2e^{-3x}.$ |
| 2. | $\dfrac{d^2y}{dx^2} + y = 0.$ | $y = c_1 \sin x + c_2 \cos x.$ |
| 3. | $\dfrac{d^2y}{dx^2} + 12y = 7\dfrac{dy}{dx}.$ | $y = c_1e^{3x} + c_2e^{4x}.$ |
| 4. | $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0.$ | $y = (c_1 + c_2x)\,e^{2x}.$ |
| 5. | $\dfrac{d^3y}{dx^3} - 4\dfrac{dy}{dx} = 0.$ | $y = c_1 + c_2e^{2x} + c_3e^{-2x}.$ |
| 6. | $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} - 8y = 0.$ | $y = c_1e^{x\sqrt{2}} + c_2e^{-x\sqrt{2}} + c_3 \sin 2x + c_4 \cos 2x.$ |
| 7. | $\dfrac{d^3s}{dt^3} - \dfrac{d^2s}{dt^2} - 6\dfrac{ds}{dt} = 0.$ | $s = c_1e^{3t} + c_2e^{-2t} + c_3.$ |
| 8. | $\dfrac{d^4\rho}{d\theta^4} - 12\dfrac{d^2\rho}{d\theta^2} + 27\rho = 0.$ | $\rho = c_1e^{3\theta} + c_2e^{-3\theta} + c_3e^{\theta\sqrt{3}} + c_4e^{-\theta\sqrt{3}}.$ |
| 9. | $\dfrac{d^2u}{dv^2} - 6\dfrac{du}{dv} + 13u = 0.$ | $u = (c_1 \sin 2v + c_2 \cos 2v)\,e^{3v}.$ |
| 10. | $\dfrac{d^4y}{dx^4} + 2n^2\dfrac{d^2y}{dx^2} + n^4y = 0.$ | $y = (c_1 + c_2x) \cos nx + (c_3 + c_4) \sin nx.$ |
| 11. | $\dfrac{d^3s}{dt^3} = s.$ | $s = c_1e^t + e^{-\frac{t}{2}}\left(c_2 \sin \dfrac{t\sqrt{3}}{2} + c_3 \cos \dfrac{t\sqrt{3}}{2}\right).$ |
| 12. | $\dfrac{d^3s}{dt^3} - 7\dfrac{ds}{dt} + 6s = 0.$ | $s = c_1e^{2t} + c_2e^t + c_3e^{3t}.$ |
| 13. | $\dfrac{d^4y}{dx^4} - 3\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0.$ | $y = c_1 + (c_2 + c_3x + c_4x^2)\,e^x.$ |

**Type II. The linear differential equation**

$$(I) \qquad \frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+p_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+p_ny=X,$$

*where $X$ is a function of $x$ alone, or constant, and $p_1$, $p_2$, $\cdots$, $p_n$ are constants.*

When $X=0$, $(I)$ reduces to $(A)$, Type I, p. 435,

$$(J) \qquad \frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+p_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+p_ny=0.$$

The complete solution of $(J)$ is called the *complementary function* of $(I)$.

Let $u$ be the complete solution of $(J)$, i.e. the complementary function of $(I)$, and $v$ any particular solution of $(I)$. Then

$$\frac{d^nv}{dx^n}+p_1\frac{d^{n-1}v}{dx^{n-1}}+p_2\frac{d^{n-2}v}{dx^{n-2}}+\cdots+p_nv=X,$$

and
$$\frac{d^nu}{dx^n}+p_1\frac{d^{n-1}u}{dx^{n-1}}+p_2\frac{d^{n-2}u}{dx^{n-2}}+\cdots+p_nu=0.$$

Adding, we get

$$\frac{d^n}{dx^n}(u+v)+p_1\frac{d^{n-1}}{dx^{n-1}}(u+v)+p_2\frac{d^{n-2}}{dx^{n-2}}(u+v)+\cdots+p_n(u+v)=X,$$

showing that $u+v$ is a solution* of $(I)$.

The complete solution of $(I)$ being $u+v$, we first find the complementary function $u$ by placing its left-hand member equal to zero and solving the resulting equation by the rule on p. 438.

To find the particular solution $v$ is a problem of considerable difficulty except in special cases. For the problems given in this book we may determine the particular solution $v$ by the following method.

Differentiate successively the given equation and obtain, either directly or by elimination, a new differential equation of a higher order of Type I. Solving this by the rule on p. 438, we get its complete solution containing the complementary function $u$ already found,† and additional terms. Determining the constants of the additional terms so as to satisfy the given differential equation, we get the particular solution $v$.

---

* In works on differential equations it is shown that $u+v$ is the complete solution.

† From the method of derivation it is obvious that every solution of the original equation must also be a solution of the derived equation.

The method will now be illustrated by means of examples.

**Note.** The solution of the auxiliary equation of the new derived differential equation is facilitated by observing that the left-hand member of that equation is exactly divisible by the left-hand member of the auxiliary equation used in finding the complementary function.

**Ex. 1.** Solve

$$(K) \qquad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = ae^{-2x}.$$

*Solution.* The complementary function $u$ of $(K)$ is the complete solution of the equation

$$(L) \qquad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0.$$

Applying the rule on p. 438, we get as the complete solution of $(L)$

$$(M) \qquad u = c_1e^x + c_2e^{-2x}.$$

Differentiating $(K)$ gives

$$(N) \qquad \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = -2ae^{-2x}.$$

Multiplying $(K)$ by 2 and adding the result to $(N)$, we get

$$(O) \qquad \frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 4y = 0,$$

a differential equation of Type I. Solving by the rule on p. 438, we get the complete solution of $(O)$ to be

$$y = c_1e^x + c_2e^{-2x} + c_3xe^{-2x},$$

or, from $(M)$,

$$y = u + c_3xe^{-2x}.$$

We now determine $c_3$ so that $c_3xe^{-2x}$ shall be a particular solution $v$ of $(K)$. Replacing $y$ in $(K)$ by $c_3xe^{-2x}$, we get

$$c_3e^{-2x}(-4+1) = ae^{-2x}.$$

$$\therefore -3c_3 = a, \text{ or, } c_3 = -\tfrac{1}{3}a.$$

Hence a particular solution of $(K)$ is

$$v = -\tfrac{1}{3}axe^{-2x},$$

and the complete solution is

$$y = u + v = c_1e^x + c_2e^{-2x} - \tfrac{1}{3}axe^{-2x}.$$

Ex. 2. Solve

$$(P) \qquad \frac{d^2y}{dx^2} + n^2y = \cos ax.$$

*Solution.* Solving

$$(Q) \qquad \frac{d^2y}{dx^2} + n^2y = 0,$$

we get the complementary function

$$(R) \qquad u = c_1\sin nx + c_2\cos nx.$$

Differentiating (*P*) twice, we get

$$(S) \qquad \frac{d^4y}{dx^4} + n^2\frac{d^2y}{dx^2} = -a^2\cos ax.$$

Multiplying (*P*) by $a^2$ and adding the result to (*S*) gives

$$(T) \qquad \frac{d^4y}{dx^4} + (n^2 + a^2)\frac{d^2y}{dx^2} + a^2n^2y = 0.$$

The complete solution of (*T*) is

$$y = c_1\sin nx + c_2\cos nx + c_3\sin ax + c_4\cos ax,$$

or,

$$y = u + c_3\sin ax + c_4\cos ax.$$

Let us now determine $c_3$ and $c_4$ so that $c_3\sin ax + c_4\cos ax$ shall be a solution of (*P*). Replacing $y$ in (*P*) by $c_3\sin ax + c_4\cos ax$, we get

$$(n^2c_4 - a^2c_4)\cos ax + (n^2c_3 - a^2c_3)\sin ax = \cos ax.$$

Equating the coefficients of like terms in this identity, we get

$$n^2c_4 - a^2c_4 = 1 \text{ and } n^2c_3 - a^2c_3 = 0,$$

or,

$$c_4 = \frac{1}{n^2 - a^2} \text{ and } c_3 = 0.$$

Hence a particular solution of (*P*) is

$$v = \frac{\cos ax}{n^2 - a^2},$$

and the complete solution is

$$y = u + v = c_1\sin nx + c_2\cos nx + \frac{\cos ax}{n^2 - a^2}.$$

## EXAMPLES

| | *Differential equations* | *Complete solutions* |
|---|---|---|
| 1. | $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = x.$ | $y = c_1e^{3x} + c_2e^{4x} + \frac{12x + 7}{144}.$ |
| 2. | $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = a.$ | $y = c_1\sin x + c_2\cos x + (c_3 + c_4x)e^x + a.$ |
| 3. | $\frac{d^2s}{dt^2} - a^2s = t + 1.$ | $s = c_1e^{at} + c_2e^{-at} - \frac{x + 1}{a^2}.$ |
| 4. | $\frac{d^3\rho}{d\theta^3} - 2\frac{d^2\rho}{d\theta^2} + \frac{d\rho}{d\theta} = e^\theta.$ | $\rho = \left(c_1 + c_2x + \frac{x^2}{2}\right)e^x + c_3.$ |
| 5. | $\frac{d^4y}{dx^4} - a^4y = x^3.$ | $y = c_1e^{ax} + c_2e^{-ax} + c_3\sin ax + c_4\cos ax - \frac{x^3}{a^4}.$ |
| 6. | $\frac{d^2s}{dt^2} + a^2s = \cos ax.$ | $y = c_1\sin ax + c_2\cos ax + \frac{x\sin ax}{2a}$ |
| 7. | $\frac{d^2s}{dt^2} - 2a\frac{ds}{dt} + a^2s = e^t.$ | $s = (c_1 + c_2t)e^{at} + \frac{e^t}{(a - 1)^2}.$ |

| | *Differential equations* | *Complete solutions* |
|---|---|---|
| **8.** | $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{nx}.$ | $y = c_1e^{2x} + c_2e^{3x} + \frac{e^{nx}}{n^2 - 5n + 6}.$ |
| **9.** | $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{nx}.$ | $y = c_1e^{x} + c_2e^{2x} + \frac{xe^{nx}}{n^2 - 3n + 2} - \frac{(2n-3)e^{nx}}{(n^2 - 3n + 2)^2}.$ |
| **10.** | $\frac{d^2s}{dt^2} - 9\frac{ds}{dt} + 20s = t^2e^{3t}.$ | $= c_1e^{4t} + c_2e^{5t} + \frac{2t^2 + 6t + 7}{4}e^{3t}.$ |
| **11.** | $\frac{d^2s}{dt^2} + 4s = t\sin^2 t.$ | $s = \left(c_1 - \frac{x^2}{16}\right)\sin 2x + \left(c_2 - \frac{x}{32}\right)\cos 2x + \frac{x}{8}.$ |

**Type III.** $$\frac{d^ny}{dx^n} = X,$$

where $X$ is a function of $x$ alone, or constant.

To solve this type of differential equations we have the following rule from Chapter XXXI, p. 392:

***Integrate n times successively. Each integration will introduce one arbitrary constant.***

Ex. 1. Solve $$\frac{d^3y}{dx^3} = xe^x.$$

*Solution.* Integrating the first time,

$$\frac{d^2y}{dx^2} = \int xe^x dx,$$

or, $$\frac{d^2y}{dx^2} = xe^x - e^x + C_1.$$ By (*A*), p. 341

Integrating the second time,

$$\frac{dy}{dx} = \int xe^x \cdot dx - \int e^x dx + \int C_1 dx,$$

$$\frac{dy}{dx} = xe^x - 2e^x + C_1x + C_2.$$

Integrating the third time,

$$y = \int xe^x dx - \int 2e^x dx + \int C_1 x dx + \int C_2 dx$$

$$= xe^x - 3e^x + \frac{C_1x^2}{2} + C_2x + C_3,$$

or, $$y = xe^x - 3e^x + c_1x^2 + c_2x + c_3.$$ ***Ans.***

**Type IV.** $$\frac{d^2y}{dx^2} = Y,$$

where $Y$ is a function of $y$ alone.

The rule for integrating this type is as follows:

**First step.** *Multiply the left-hand member by the factor*

$$2\frac{dy}{dx}dx,$$

*and the right-hand member by the equivalent factor*

$$2\,dy,$$

*and integrate. The integral of the left-hand member will be*[*]

$$\left(\frac{dy}{dx}\right)^2.$$

**Second step.** *Extract the square root of both members, separate the variables, and integrate again.*[†]

Ex. 1. Solve $$\frac{d^2y}{dx^2} + a^2y = 0.$$

*Solution.* Here $\frac{d^2y}{dx^2} = -a^2y$, hence of Type IV.

*First step.* Multiplying the left-hand member by $2\frac{dy}{dx}dx$ and the right-hand member by $2\,dy$, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}dx = -2a^2ydy.$$

Integrating, $$\left(\frac{dy}{dx}\right)^2 = -a^2y^2 + C_1.$$

*Second step.* $$\frac{dy}{dx} = \sqrt{C_1 - a^2y^2},$$

taking the positive sign of the radical. Separating the variables, we get

$$\frac{dy}{\sqrt{C_1 - a^2y^2}} = dx.$$

Integrating, $$\frac{1}{a}\text{arc sin}\frac{ay}{\sqrt{C_1}} = x + C_2,$$

or, $$\text{arc sin}\frac{ay}{\sqrt{C_1}} = ax + aC_2.$$

This is the same as $$\frac{ay}{\sqrt{C_1}} = \sin(ax + aC_2)$$

$$= \sin ax \cos aC_2 + \cos ax \sin aC_2, \qquad 31, \text{ p. } 2$$

or, $$y = \frac{\sqrt{C_1}}{a}\cos aC_2 \cdot \sin ax + \frac{\sqrt{C_1}}{a}\sin aC_2 \cdot \cos ax$$

$$= c_1 \sin ax + c_2 \cos ax. \quad \textit{Ans.}$$

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[*] Since $d\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx}\frac{d^2y}{dx^2}dx.$

[†] Each integration introduces an arbitrary constant.

**EXAMPLES**

| *Differential equations* | *Solutions* |
|---|---|
| 1. $\dfrac{d^3y}{dx^3} = x^2 - 2\cos x.$ | $y = \dfrac{x^5}{60} + 2\sin x + c_1x^2 + c_2x + c_3.$ |
| 2. $v\dfrac{d^3u}{dv^3} = 2.$ | $u = v^2 \log v + c_1v^2 + c_2v + c_3.$ |
| 3. $\dfrac{d^3\rho}{d\theta^3} = \sin^3\theta.$ | $\rho = -\dfrac{\cos^3\theta}{27} + \dfrac{7\cos\theta}{9} + c_1\theta^2 + c_2\theta + c_3.$ |
| 4. $\dfrac{d^2s}{dt^2} = f\sin nt.$ | $s = -\dfrac{f}{n^2}\sin nt + c_1t + c_2.$ |
| 5. $\dfrac{d^2s}{dt^2} = g.$ | $s = \frac{1}{2}gt^2 + c_1t + c_2.$ |
| 6. $\dfrac{d^ny}{dx^n} = x^m.$ | $y = \dfrac{\underline{\lvert m}\, x^{m+n}}{\underline{\lvert m+n}} + c_1x^{n-1} + \cdots + c_{n-1}x + c_n.$ |
| 7. $\dfrac{d^2y}{dx^2} = a^2y.$ | $ax = \log(y + \sqrt{y^2 + c_1}) + c_2,$ or<br>$y = c_1e^{ax} + c_2e^{-ax}.$ |
| 8. $\dfrac{d^2s}{dt^2} = \dfrac{1}{\sqrt{as}}.$ | $3t = 2a^{\frac{1}{4}}(s^{\frac{1}{2}} - 2c_1)(s^{\frac{1}{2}} + c_1)^{\frac{1}{2}} + c_2.$ |
| 9. $\dfrac{d^2y}{dt^2} = \dfrac{a}{y^3}.$ | $(c_1t + c_2)^2 + a = c_1y^2.$ |
| 10. $\dfrac{d^2x}{dt^2} = e^{nx}.$ | $t\sqrt{2n} = c_1 \log \dfrac{\sqrt{c_1^2e^{nx} + 1} - 1}{\sqrt{c_1^2e^{nx} + 1} + 1} + c_2.$ |

11. $\dfrac{d^2s}{dt^2} = -\dfrac{k}{s^2}.$ Find $t$, having given that $\dfrac{dx}{dt} = 0$ and $x = a$, when $t = 0$.

$$\textit{Ans.}\quad t = \sqrt{\frac{a}{2k}}\left\{\frac{a}{2}\left(\text{arc vers}\,\frac{2s}{a} - \pi\right) - \sqrt{as - s^2}\right\}.$$

# CHAPTER XXXIII

## INTEGRAPH. TABLE OF INTEGRALS

**245. Mechanical integration.** We have seen that the determination of the area bounded by a curve $C$ whose equation is

$$y = f(x)$$

and the evaluation of the definite integral

$$\int_C f(x)\,dx$$

are equivalent problems (§ 209, p. 357).

Hitherto we have regarded the relation between the variables $x$ and $y$ as given by analytical formulas and have applied analytic methods in obtaining the integrals required. If, however, the relation between the variables is given, not analytically, but as frequently is the case in physical investigations, graphically, i.e. by a curve,[*] the analytic method is inapplicable unless the exact or approximate equation of the curve can be obtained. It is, however, possible to determine the area bounded by a curve, whether we know its equation or not, by means of mechanical devices. We shall consider the construction theory and the use of one such device, namely the Integraph, invented by Abdank-Abakanowicz.[†] Before proceeding with the discussion of this machine it is necessary to take up the study of *integral curves.*

**246. Integral curves.** If $F(x)$ and $f(x)$ are two functions so related that

$$(A) \qquad \frac{d}{dx}F(x) = f(x),$$

then the curve

$$(B) \qquad y = F(x)$$

---

[*] For instance the record made by a registering thermometer, a steam-engine indicator, or by certain testing machines.

[†] See *Les Integraphes; la courbe intégrale et ses applications*, by Abdank-Abakanowicz, Paris, 1889.

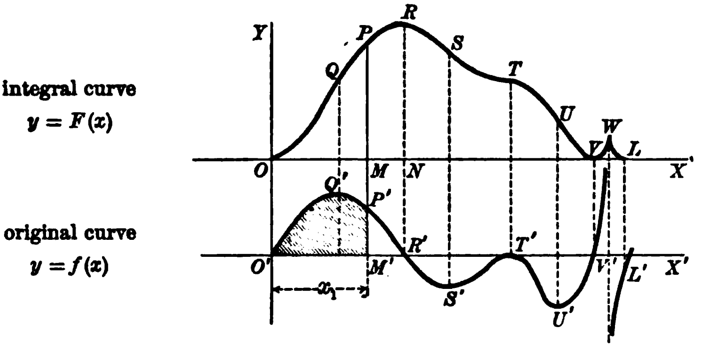
is called an *integral curve* of the curve

(C) $$y = f(x).^*$$

The name *integral curve* is due to the fact that from (C) it is seen that the same relation between the functions may be expressed as follows:

(D) $$\int_0^x f(x)\,dx = F(x). \qquad F(0) = 0$$

Let us draw an original curve and a corresponding integral curve in such a way as easily to compare their corresponding points.



To find an expression for the shaded portion (O'M'P') of the area under the original curve we substitute in (A), p. 371, giving

$$\text{area } O'M'P' = \int_0^{x_1} f(x)dx.$$

But from (D) this becomes

$$\text{area } O'M'P' = \int_0^{x_1} f(x)\,dx = [F(x)]_{x=x_1} = F(x_1) = MP.^\dagger$$

**Theorem.** *For the same abscissa $x_1$, the number giving the length of the ordinate of the integral curve (B) is the same as the number that gives the area between the original curve, the axes, and the ordinate corresponding to this abscissa.*

---

[*] This curve is sometimes called the *original curve*.

[†] When $x_1 = O'R'$, the positive area $O'M'R'P'$ is represented by the maximum ordinate $NR$. To the right of $R'$ the area is below the axis of $X$ and therefore negative; consequently the ordinates of the integral curve, which represent the algebraic sum of the areas inclosed, will decrease in passing from $R'$ to $T'$.

The most general integral curve is of the form

$$y = F(x) + C,$$

in which case the *difference* of the ordinates for $x = 0$ and $x = x_1$ gives the area under the original curve. In the integral curve drawn $C = F(0) = 0$, i.e. the general integral curve is obtained if this integral curve be displaced the distance $C$ parallel to $OY$.

The student should also observe that

(a) For the same abscissa $x_1$, the number giving the slope of the integral curve is the same as the number giving the length of the corresponding ordinate of the original curve [from $(C)$]. Hence $(C)$ is sometimes called the *curve of slopes* of $(E)$. In the figure we see that at points $O$, $R$, $T$, $V$, where the integral curve is parallel to $OX$, the corresponding points $O'$, $R'$, $T'$, $V'$ on the original curve have zero ordinates, and corresponding to the point $W$ the original curve is discontinuous.

(b) Corresponding to points of inflection $Q$, $S$, $U$ on the integral curve we have maximum or minimum ordinates to the original curve.

For example, since

$$\frac{d}{dx}\left(\frac{x^3}{9}\right)=\frac{x^2}{3},$$

it follows that

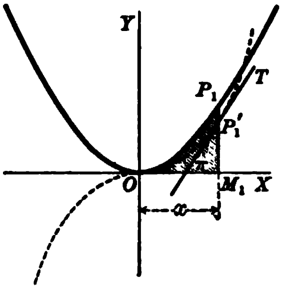
$$(E) \qquad y=\frac{x^3}{9}$$

is an integral curve of the parabola

$$(F) \qquad y=\frac{x^2}{3}.$$

Since from $(F)$

$$\text{area } OM_1P_1=\int_0^{x_1}\frac{x^2}{3}dx=\frac{x_1^3}{9},$$



and from $(E)$

$$M_1P_1'=\frac{x_1^3}{9},$$

it is seen that $\frac{x_1^3}{9}$ indicates the number of linear units in the ordinates $M^1P^{1\prime}$, and also the number of units of area in the shaded area $OM_1P_1$.

Also since from $(E)$ $\frac{dy}{dx}=\frac{x^2}{3}$, or $\tan\tau=\frac{x_1^2}{3}$,

and from $(F)$

$$M_1P_1=\frac{x_1^2}{3},$$

it is seen that the same number $\frac{x_1^2}{3}$ indicates the length of ordinate $M_1P_1$ and the slope of the tangent at $P_1'$.

Evidently the origin is a point of inflection of the integral curve and a point with minimum ordinate on the original curve.

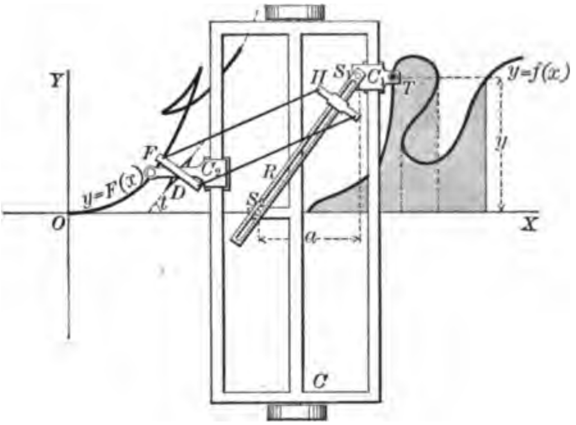
**247. The integraph.** The theory of this instrument is exceedingly simple and depends on the relation between the given curve and a corresponding integral curve.

The instrument is constructed as follows. A rectangular carriage $C$ moves on rollers over the plane in a direction parallel to the axis of $X$ of the curve

$$y=f(x).$$

Two sides of the carriage are parallel to the axis of $X$; the other two, of course, perpendicular to it. Along one of these perpendicular sides moves a small carriage $C_1$ bearing the tracing point $T$, and along the other a small carriage $C_2$ bearing a frame $F$ which can revolve about an axis perpendicular to the surface, and carries the sharp-edged disk $D$ to the plane of which it is perpendicular. A stud $S_1$ is fixed in the carriage $C_1$ so as to be at the same distance from the axis of $X$ as is the tracing point $T$. A second stud $S_2$ is set in a crossbar of the main carriage $C$

so as to be on the axis of $X$. A split ruler $R$ joins these two studs and slides upon them. A crosshead $H$ slides upon this ruler and is joined to the frame $F$ by a parallelogram.



The essential part of the instrument consists of the sharp-edged disk $D$, which moves under pressure over a smooth plane surface (paper). This disk will not slide, and hence as it rolls must always move along a path the tangent to which at every point is the trace of the plane of the disk. If now this disk is caused to move, it is evident from the figure that the construction of the machine insures that the plane of the disk $D$ shall be parallel to the ruler $R$. But if $a$ is the distance between the ordinates through the studs $S_1$, $S_2$, and $\tau$ is the angle made by $R$ (and therefore also plane of disk) with the axis of $X$, we have

$$(A) \qquad \tan\tau = \frac{y}{a};$$

and if

$$y' = F(x')$$

is the curve traced by the point of contact of the disk, we have

$$(B) \qquad \tan\tau = \frac{dy'}{dx}.^*$$

Comparing $(A)$ and $(B)$,

$$\frac{dy'}{dx} = \frac{y}{a}, \text{ or,}$$

$$(C) \qquad y' = \frac{1}{a}\int y\,dx = \frac{1}{a}\int f(x)\,dx = F'(x).^\dagger$$

That is (dropping the primes), the curve

$$y = F(x)$$

is an *integral curve* of the curve

$$(D) \qquad y = \frac{1}{a}f(x).$$

The factor $\frac{1}{a}$ evidently fixes merely the *scale* to which the integral curve is drawn, and does not affect its *form*.

A pencil or pen is attached to the carriage $C_2$ in order to draw the curve $y = F(x)$. Displacing the disk $D$ before tracing the original curve is equivalent to changing the constant of integration.

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* Since $x = x' + d$, where $d =$ width of machine, and therefore $\frac{dy'}{dx'} = \frac{dy'}{dx} \cdot \frac{dx}{dx'} = \frac{dy}{dx'}$.

† It is assumed that the instrument is so constructed that the abscissas of any two corresponding points of the two curves differ only by a constant; hence $x$ is a function of $x'$.

**248. Integrals for reference.** Following is a table of integrals for reference. In going over the subject of Integral Calculus for the first time, the student is advised to use this table sparingly, if at all. As soon as the derivation of these integrals is thoroughly understood, the table may be properly used for saving time and labor in the solution of practical problems.

### SOME ELEMENTARY FORMS

1. $\int (du \pm dv \pm dw \pm \cdots) = \int du \pm \int dv \pm \int dw \pm \cdots.$

2. $\int a\,dv = a \int dv.$

3. $\int df(x) = \int f'(x)\,dx = f(x) + C.$

4. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C,\ n \pm 1.$

5. $\int \dfrac{dx}{x} = \log x + C.$

### FORMS CONTAINING INTEGRAL POWERS OF $a + bx$

6. $\int \dfrac{dx}{a+bx} = \dfrac{1}{b} \log (a+bx) + C.$

7. $\int (a+bx)^n dx = \dfrac{(a+bx)^{n+1}}{b(n+1)} + C,\ n \pm 1.$

8. $\int F(x, a+bx)\,dx.$ Try one of the substitutions, $z = a + bx$, $xz = a + bx$.

9. $\int \dfrac{x\,dx}{a+bx} = \dfrac{1}{b^2}[a + bx - a \log (a+bx)] + C.$

10. $\int \dfrac{x^2 dx}{a+bx} = \dfrac{1}{b^3}[\tfrac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2 \log (a+bx)] + C.$

11. $\int \dfrac{dx}{x(a+bx)} = -\dfrac{1}{a} \log \dfrac{a+bx}{x} + C.$

12. $\int \dfrac{dx}{x^2(a+bx)} = -\dfrac{1}{ax} + \dfrac{b}{a^2} \log \dfrac{a+bx}{x} + C.$

13. $\int \dfrac{x\,dx}{(a+bx)^2} = \dfrac{1}{b^2}\left[\log (a+bx) + \dfrac{a}{a+bx}\right] + C.$

14. $\int \dfrac{x^2 dx}{(a+bx)^2} = \dfrac{1}{b^3}\left[a + bx - 2a \log (a+bx) - \dfrac{a^2}{a+bx}\right] + C.$

15. $\displaystyle\int\frac{dx}{x(a+bx)^2}=\frac{1}{a(a+bx)}-\frac{1}{a^2}\log\frac{a+bx}{x}+C.$

16. $\displaystyle\int\frac{x\,dx}{(a+bx)^3}=\frac{1}{b^2}\left[-\frac{1}{a+bx}+\frac{a}{2(a+bx)^2}\right]+C.$

### FORMS CONTAINING $a^2+x^2$, $a^2-x^2$, $a+bx^n$, $a+bx^2$

17. $\displaystyle\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C;\quad\int\frac{dx}{1+x^2}=\tan^{-1}x+C.$

18. $\displaystyle\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+x}{a-x}+C;\quad\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}+C.$

19. $\displaystyle\int\frac{dx}{a+bx^2}=\frac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\frac{b}{a}}+C,$ when $a>0$ and $b>0$.

20. $\displaystyle\int\frac{dx}{a^2-b^2x^2}=\frac{1}{2ab}\log\frac{a+bx}{a-bx}+C.$

21. $\displaystyle\int x^m(a+bx^n)^p\,dx$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)}-\frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p\,dx.$$

22. $\displaystyle\int x^m(a+bx^n)^p\,dx=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}+\frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$

23. $\displaystyle\int\frac{dx}{x^m(a+bx^n)^p}$

$$=-\frac{1}{(m-1)ax^{m-1}(a+bx^n)^{p-1}}-\frac{(m-n+np-1)b}{(m-1)a}\int\frac{dx}{x^{m-n}(a+bx^n)^p}.$$

24. $\displaystyle\int\frac{dx}{x^m(a+bx^n)^p}$

$$=\frac{1}{an(p-1)x^{m-1}(a+bx^n)^{p-1}}+\frac{m-n+np-1}{an(p-1)}\int\frac{dx}{x^m(a+bx^n)^{p-1}}.$$

25. $\displaystyle\int\frac{(a+bx^n)^p\,dx}{x^m}=-\frac{(a+bx^n)^{p+1}}{a(m-1)x^{m-1}}-\frac{b(m-n-np-1)}{a(m-1)}\int\frac{(a+bx^n)^p\,dx}{x^{m-n}}.$

26. $\displaystyle\int\frac{(a+bx^n)^p\,dx}{x^m}=\frac{(a+bx^n)^p}{(np-m+1)x^{m-1}}+\frac{anp}{np-m+1}\int\frac{(a+bx^n)^{p-1}dx}{x^m}.$

27. $\displaystyle\int\frac{x^m dx}{(a+bx^n)^p}=\frac{x^{m-n+1}}{b(m-np+1)(a+bx^n)^{p-1}}-\frac{a(m-n+1)}{b(m-np+1)}\int\frac{x^{m-n}dx}{(a+bx^n)^p}.$

28. $\displaystyle\int\frac{x^m dx}{(a+bx^n)^p}=\frac{x^{m+1}}{an(p-1)(a+bx^n)^{p-1}}-\frac{m+n-np+1}{an(p-1)}\int\frac{x^m dx}{(a+bx^n)^{p-1}}.$

29. $\displaystyle\int\frac{dx}{(a^2+x^2)^n}=\frac{1}{2(n-1)a^2}\left[\frac{x}{(a^2+x^2)^{n-1}}+(2n-3)\int\frac{dx}{(a^2+x^2)^{n-1}}\right].$

30. $$\int\frac{dx}{(a+bx^2)^n}=\frac{1}{2(n-1)a}\left[\frac{x}{(a+bx^2)^{n-1}}+(2n-3)\int\frac{dx}{(a+bx^2)^{n-1}}\right].$$

31. $$\int\frac{xdx}{(a+bx^2)^n}=\frac{1}{2}\int\frac{dz}{(a+bz)^n},\text{ where } z=x^2.$$

32. $$\int\frac{x^2dx}{(a+bx^2)^n}=\frac{-x}{2b(n-1)(a+bx^2)^{n-1}}+\frac{1}{2b(n-1)}\int\frac{dx}{(a+bx^2)^{n-1}}.$$

33. $$\int\frac{dx}{x(a+bx^n)}=\frac{1}{an}\log\frac{x^n}{a+bx^n}+C.$$

34. $$\int\frac{dx}{x^2(a+bx^2)^n}=\frac{1}{a}\int\frac{dx}{x^2(a+bx^2)^{n-1}}-\frac{b}{a}\int\frac{dx}{(a+bx^2)^n}.$$

35. $$\int\frac{xdx}{a+bx^2}=\frac{1}{2b}\log\left(x^2+\frac{a}{b}\right)+C.$$

36. $$\int\frac{x^2dx}{a+bx^2}=\frac{x}{b}-\frac{a}{b}\int\frac{dx}{a+bx^2}.$$

37. $$\int\frac{dx}{x(a+bx^2)}=\frac{1}{2a}\log\frac{x^2}{a+bx^2}+C.$$

38. $$\int\frac{dx}{x^2(a+bx^2)}=-\frac{1}{ax}-\frac{b}{a}\int\frac{dx}{a+bx^2}.$$

39. $$\int\frac{dx}{(a+bx^2)^2}=\frac{x}{2a(a+bx^2)}+\frac{1}{2a}\int\frac{dx}{a+bx^2}.$$

### FORMS CONTAINING $\sqrt{a+bx}$

40. $$\int x\sqrt{a+bx}\,dx=-\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}+C.$$

41. $$\int x^2\sqrt{a+bx}\,dx=\frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}+C.$$

42. $$\int\frac{xdx}{\sqrt{a+bx}}=-\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}+C.$$

43. $$\int\frac{x^2dx}{\sqrt{a+bx}}=\frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}+C.$$

44. $$\int\frac{dx}{x\sqrt{a+bx}}=\frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}+C,\text{ for } a>0.$$

45. $$\int\frac{dx}{x\sqrt{a+bx}}=\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}+C,\text{ for } a<0.$$

46. $$\int\frac{dx}{x^2\sqrt{a+bx}}=\frac{-\sqrt{a+bx}}{ax}-\frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}.$$

47. $$\int\frac{\sqrt{a+bx}\,dx}{x}=2\sqrt{a+bx}+a\int\frac{dx}{x\sqrt{a+bx}}.$$

### FORMS CONTAINING $\sqrt{x^2+a^2}$

48. $$\int(x^2+a^2)^{\frac{1}{2}}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log(x+\sqrt{x^2+a^2})+C.$$

49. $$\int(x^2+a^2)^{\frac{3}{2}}dx=\frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2}+\frac{3a^4}{8}\log(x+\sqrt{x^2+a^2})+C.$$

50. $$\int(x^2+a^2)^{\frac{n}{2}}dx=\frac{x(x^2+a^2)^{\frac{n}{2}}}{n+1}+\frac{na^2}{n+1}\int(x^2+a^2)^{\frac{n}{2}-1}dx.$$

51. $$\int x(x^2+a^2)^{\frac{n}{2}}dx=\frac{(x^2+a^2)^{\frac{n+2}{2}}}{n+2}+C.$$

52. $$\int x^2(x^2+a^2)^{\frac{1}{2}}dx=\frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2}-\frac{a^4}{8}\log(x+\sqrt{x^2+a^2})+C.$$

53. $$\int\frac{dx}{(x^2+a^2)^{\frac{1}{2}}}=\log(x+\sqrt{x^2+a^2})+C.$$

54. $$\int\frac{dx}{(x^2+a^2)^{\frac{3}{2}}}=\frac{x}{a^2\sqrt{x^2+a^2}}+C.$$

55. $$\int\frac{xdx}{(x^2+a^2)^{\frac{1}{2}}}=\sqrt{x^2+a^2}+C.$$

56. $$\int\frac{x^2dx}{(x^2+a^2)^{\frac{1}{2}}}=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\log(x+\sqrt{x^2+a^2})+C.$$

57. $$\int\frac{x^2dx}{(x^2+a^2)^{\frac{3}{2}}}=-\frac{x}{\sqrt{x^2+a^2}}+\log(x+\sqrt{x^2+a^2})+C.$$

58. $$\int\frac{dx}{x(x^2+a^2)^{\frac{1}{2}}}=\frac{1}{a}\log\frac{x}{a+\sqrt{x^2+a^2}}+C.$$

59. $$\int\frac{dx}{x^2(x^2+a^2)^{\frac{1}{2}}}=-\frac{\sqrt{x^2+a^2}}{a^2x}+C.$$

60. $$\int\frac{dx}{x^3(x^2+a^2)^{\frac{1}{2}}}=-\frac{\sqrt{x^2+a^2}}{2a^2x^2}+\frac{1}{2a^3}\log\frac{a+\sqrt{x^2+a^2}}{x}+C.$$

61. $$\int\frac{(x^2+a^2)^{\frac{1}{2}}dx}{x}=\sqrt{a^2+x^2}-a\log\frac{a+\sqrt{a^2+x^2}}{x}+C.$$

62. $$\int\frac{(x^2+a^2)^{\frac{1}{2}}dx}{x^2}=-\frac{\sqrt{x^2+a^2}}{x}+\log(x+\sqrt{x^2+a^2})+C.$$

### FORMS CONTAINING $\sqrt{x^2 - a^2}$

63. $\int (x^2 - a^2)^{\frac{1}{2}} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log (x + \sqrt{x^2 - a^2}) + C.$

64. $\int (x^2 - a^2)^{\frac{3}{2}} dx = \frac{x}{8} (2 x^2 - 5 a^2) \sqrt{x^2 - a^2} + \frac{3 a^4}{8} \log (x + \sqrt{x^2 - a^2}) + C.$

65. $\int (x^2 - a^2)^{\frac{n}{2}} dx = \frac{x (x^2 - a^2)^{\frac{n}{2}}}{n + 1} - \frac{na^2}{n + 1} \int (x^2 + a^2)^{\frac{n}{2} - 1} dx.$

66. $\int x (x^2 - a^2)^{\frac{n}{2}} dx = \frac{(x^2 - a^2)^{\frac{n+2}{2}}}{n + 2} + C.$

67. $\int x^2 (x^2 - a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2 x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \log (x + \sqrt{x^2 - a^2}) + C.$

68. $\int \frac{dx}{(x^2 - a^2)^{\frac{1}{2}}} = \log (x + \sqrt{x^2 - a^2}) + C.$

69. $\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}} = - \frac{x}{a^2 \sqrt{x^2 - a^2}} + C.$

70. $\int \frac{xdx}{(x^2 - a^2)^{\frac{1}{2}}} = \sqrt{x^2 - a^2} + C.$

71. $\int \frac{x^2dx}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{x}{2} \sqrt{x^2 - a^2} + \frac{a^2}{2} \log (x + \sqrt{x^2 - a^2}) + C.$

72. $\int \frac{x^2dx}{(x^2 - a^2)^{\frac{3}{2}}} = - \frac{x}{\sqrt{x^2 - a^2}} + \log (x + \sqrt{x^2 - a^2}) + C.$

73. $\int \frac{dx}{x (x^2 - a^2)^{\frac{1}{2}}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C; \int \frac{dx}{x \sqrt{x^2 - 1}} = \sec^{-1} x + C.$

74. $\int \frac{dx}{x^2 (x^2 - a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2 - a^2}}{a^2x} + C.$

75. $\int \frac{dx}{x^3 (x^2 - a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2 - a^2}}{2 a^2x^2} + \frac{1}{2 a^3} \sec^{-1} \frac{x}{a} + C.$

76. $\int \frac{(x^2 - a^2)^{\frac{1}{2}} dx}{x} = \sqrt{x^2 - a^2} - a \cos^{-1} \frac{a}{x} + C.$

77. $\int \frac{(x^2 - a^2)^{\frac{1}{2}} dx}{x^2} = - \frac{\sqrt{x^2 - a^2}}{x} + \log (x + \sqrt{x^2 - a^2}) + C.$

### FORMS CONTAINING $\sqrt{a^2 - x^2}$.

78. $\int (a^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C.$

79. $\int (a^2 - x^2)^{\frac{3}{2}} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{a} + C.$

80. $\int (a^2 - x^2)^{\frac{n}{2}} dx = \frac{x(a^2 - x^2)^{\frac{n}{2}}}{n+1} + \frac{a^2 n}{n+1} \int (a^2 - x^2)^{\frac{n}{2}-1} dx.$

81. $\int x (a^2 - x^2)^{\frac{n}{2}} dx = -\frac{(a^2 - x^2)^{\frac{n+2}{2}}}{n+2} + C.$

82. $\int x^2 (a^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1} \frac{x}{a} + C.$

83. $\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \sin^{-1} \frac{x}{a}; \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x.$

84. $\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C.$

85. $\int \frac{x dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\sqrt{a^2 - x^2} + C.$

86. $\int \frac{x^2 dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C.$

87. $\int \frac{x^2 dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a} + C.$

88. $\int \frac{x^m dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\frac{x^{m-1}}{m} \sqrt{a^2 - x^2} + \frac{(m-1)a^2}{m} \int \frac{x^{m-2}}{(a^2 - x^2)^{\frac{1}{2}}} dx.$

89. $\int \frac{dx}{x (a^2 - x^2)^{\frac{1}{2}}} = \frac{1}{a} \log \frac{x}{a + \sqrt{a^2 - x^2}} + C.$

90. $\int \frac{dx}{x^2 (a^2 - x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$

91. $\int \frac{dx}{x^3 (a^2 - x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2 - x^2}}{2 a^2 x^2} + \frac{1}{2 a^3} \log \frac{x}{a + \sqrt{a^2 - x^2}} + C.$

92. $\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x} dx = \sqrt{a^2 - x^2} - a \log \frac{a + \sqrt{a^2 - x^2}}{x} + C.$

93. $\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a} + C.$

### FORMS CONTAINING $\sqrt{2ax - x^2}$, $\sqrt{2ax + x^2}$

94. $\int \sqrt{2ax - x^2}\,dx = \frac{x-a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a} + C.$

95. $\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{vers}^{-1}\frac{x}{a}; \int \frac{dx}{\sqrt{2x - x^2}} = \operatorname{vers}^{-1}x + C.$

96. $\int x^m\sqrt{2ax - x^2}\,dx = -\frac{x^{m-1}(2ax - x^2)^{\frac{3}{2}}}{m+2} + \frac{(2m+1)a}{m+2}\int x^{m-1}\sqrt{2ax - x^2}\,dx.$

97. $\int \frac{dx}{x^m\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{(2m-1)ax^m} + \frac{m-1}{(2m-1)a}\int \frac{dx}{x^{m-1}\sqrt{2ax - x^2}}.$

98. $\int \frac{x^m dx}{\sqrt{2ax - x^2}} = -\frac{x^{m-1}\sqrt{2ax - x^2}}{m} + \frac{(2m-1)a}{m}\int \frac{x^{m-1}dx}{\sqrt{2ax - x^2}}.$

99. $\int \frac{\sqrt{2ax - x^2}}{x^m}dx = -\frac{(2ax - x^2)^{\frac{3}{2}}}{(2m-3)ax^m} + \frac{m-3}{(2m-3)a}\int \frac{\sqrt{2ax - x^2}}{x^{m-1}}dx.$

100. $\int x\sqrt{2ax - x^2}\,dx = -\frac{3a^2 + ax - 2x^2}{6}\sqrt{2ax - x^2} + \frac{a^3}{2}\operatorname{vers}^{-1}\frac{x}{a}.$

101. $\int \frac{dx}{x\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax} + C.$

102. $\int \frac{xdx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a\operatorname{vers}^{-1}\frac{x}{a} + C.$

103. $\int \frac{x^2dx}{\sqrt{2ax - x^2}} = -\frac{x+3a}{2}\sqrt{2ax - x^2} + \frac{3}{2}a^2\operatorname{vers}^{-1}\frac{x}{a} + C.$

104. $\int \frac{\sqrt{2ax - x^2}}{x}dx = \sqrt{2ax - x^2} + a\operatorname{vers}^{-1}\frac{x}{a} + C.$

105. $\int \frac{\sqrt{2ax - x^2}}{x^2}dx = -\frac{2\sqrt{2ax - x^2}}{x} - \operatorname{vers}^{-1}\frac{x}{a} + C.$

106. $\int \frac{\sqrt{2ax - x^2}}{x^3}dx = -\frac{(2ax - x^2)^{\frac{3}{2}}}{3ax^3} + C.$

107. $\int \frac{dx}{(2ax - x^2)^{\frac{3}{2}}} = \frac{x-a}{a^2\sqrt{2ax - x^2}} + C.$

108. $\int \frac{xdx}{(2ax - x^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{2ax - x^2}} + C.$

109. $\int F(x, \sqrt{2ax - x^2})\,dx = \int F(z + a, \sqrt{a^2 - z^2})\,dz$, where $z = x - a$.

**110.** $\int \frac{dx}{\sqrt{2ax + x^2}} = \log(x + a + \sqrt{2ax + x^2}) + C.$

**111.** $\int F(x, \sqrt{2ax + x^2})\,dx = \int F(z - a, \sqrt{z^2 - a^2})\,dz$, where $z = x + a$.

### Forms containing $a + bx \pm cx^2$

**112.** $\int \frac{dx}{a + bx + cx^2} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \frac{2cx + b}{\sqrt{4ac - b^2}} + C$, when $b^2 < 4ac$.

**113.** $\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4ac}} \log \frac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}} + C$, when $b^2 > 4ac$.

**114.** $\int \frac{dx}{a + bx - cx^2} = \frac{1}{\sqrt{b^2 + 4ac}} \log \frac{\sqrt{b^2 + 4ac} + 2cx - b}{\sqrt{b^2 + 4ac} - 2cx + b} + C.$

**115.** $\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \log(2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}) + C.$

**116.** $\int \sqrt{a + bx + cx^2}\,dx$

$$= \frac{2cx + b}{4c}\sqrt{a + bx + cx^2} - \frac{b^2 - 4ac}{8c^{\frac{3}{2}}} \log(2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}) + C.$$

**117.** $\int \frac{dx}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2cx - b}{\sqrt{b^2 + 4ac}} + C.$

**118.** $\int \sqrt{a + bx - cx^2}\,dx = \frac{2cx - b}{4c}\sqrt{a + bx - cx^2} + \frac{b^2 + 4ac}{8c^{\frac{3}{2}}} \sin^{-1} \frac{2cx - b}{\sqrt{b^2 + 4ac}} + C.$

**119.** $\int \frac{x\,dx}{\sqrt{a + bx + cx^2}} = \frac{\sqrt{a + bx + cx^2}}{c} - \frac{b}{2c^{\frac{3}{2}}} \log(2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}) + C.$

**120.** $\int \frac{x\,dx}{\sqrt{a + bx - cx^2}} = -\frac{\sqrt{a + bx - cx^2}}{c} + \frac{b}{2c^{\frac{3}{2}}} \sin^{-1} \frac{2cx - b}{\sqrt{b^2 + 4ac}} + C.$

### Other Algebraic Forms

**121.** $\int \sqrt{\frac{a + x}{b + x}}\,dx = \sqrt{(a + x)(b + x)} + (a - b) \log(\sqrt{a + x} + \sqrt{b + x}) + C.$

**122.** $\int \sqrt{\frac{a - x}{b + x}}\,dx = \sqrt{(a - x)(b + x)} + (a + b) \sin^{-1} \sqrt{\frac{x + b}{a + b}} + C.$

**123.** $\int \sqrt{\frac{a + x}{b - x}}\,dx = -\sqrt{(a + x)(b - x)} - (a + b) \sin^{-1} \sqrt{\frac{b - x}{a + b}} + C.$

124. $\int \sqrt{\frac{1+x}{1-x}}\,dx = -\sqrt{1-x^2} + \sin^{-1}x + C.$

125. $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C.$

### EXPONENTIAL AND TRIGONOMETRIC FORMS

126. $\int a^x dx = \frac{a^x}{\log a} + C.$

127. $\int e^x dx = e^x + C.$

128. $\int e^{ax} dx = \frac{e^{ax}}{a} + C.$

129. $\int \sin x dx = -\cos x + C.$

130. $\int \cos x dx = \sin x + C.$

131. $\int \tan x dx = \log \sec x = -\log \cos x + C.$

132. $\int \cot x dx = \log \sin x + C.$

133. $\int \sec x dx = \int \frac{dx}{\cos x} = \log(\sec x + \tan x) = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C.$

134. $\int \operatorname{cosec} x dx = \int \frac{dx}{\sin x} = \log(\operatorname{cosec} x - \cot x) = \log \tan \frac{x}{2} + C.$

135. $\int \sec^2 x dx = \tan x + C.$

136. $\int \operatorname{cosec}^2 x dx = -\cot x + C.$

137. $\int \sec x \tan x dx = \sec x + C.$

138. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C.$

139. $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C.$

140. $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C.$

141. $\int \sin^n x dx = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx.$

142. $\int \cos^n x dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x dx.$

143. $\int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2}x}.$

144. $\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}x}.$

145. $\int \cos^m x \sin^n x dx = \frac{\cos^{m-1}x \sin^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2}x \sin^n x dx.$

146. $\int \cos^m x \sin^n x dx = -\frac{\sin^{n-1}x \cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x dx.$

147. $\int \frac{dx}{\sin^m x \cos^n x} = \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1}x \cos^{n-1}x} + \frac{m+n-2}{n-1}\int \frac{dx}{\sin^m x \cos^{n-2}x}.$

148. $$\int\frac{dx}{\sin^m x\cos^n x}=-\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}x\cos^{n-1}x}+\frac{m+n-2}{m-1}\int\frac{dx}{\sin^{m-2}x\cos^n x}.$$

149. $$\int\frac{\cos^m x\,dx}{\sin^n x}=-\frac{\cos^{m+1}x}{(n-1)\sin^{n-1}x}-\frac{m-n+2}{n-1}\int\frac{\cos^m x\,dx}{\sin^{n-2}x}.$$

150. $$\int\frac{\cos^m x\,dx}{\sin^n x}=\frac{\cos^{m-1}x}{(m-n)\sin^{n-1}x}+\frac{m-1}{m-n}\int\frac{\cos^{m-2}x\,dx}{\sin^n x}.$$

151. $$\int\sin x\cos^n x\,dx=-\frac{\cos^{n+1}x}{n+1}+C.$$

152. $$\int\sin^n x\cos x\,dx=\frac{\sin^{n+1}x}{n+1}+C.$$

153. $$\int\tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}x\,dx+C.$$

154. $$\int\cot^n x\,dx=-\frac{\cot^{n-1}x}{n-1}-\int\cot^{n-2}x\,dx+C.$$

155. $$\int\sin mx\sin nx\,dx=-\frac{\sin(m+n)x}{2(m+n)}+\frac{\sin(m-n)x}{2(m-n)}+C.$$

156. $$\int\cos mx\cos nx\,dx=\frac{\sin(m+n)x}{2(m+n)}+\frac{\sin(m-n)x}{2(m-n)}+C.$$

157. $$\int\sin mx\cos nx\,dx=-\frac{\cos(m+n)x}{2(m+n)}-\frac{\cos(m-n)x}{2(m-n)}+C.$$

158. $$\int\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)+C,\text{ when } a>b.$$

159. $$\int\frac{dx}{a+b\cos x}=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}}+C,\text{ when } a<b.$$

160. $$\int\frac{dx}{a+b\sin x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C,\text{ when } a>b.$$

161. $$\int\frac{dx}{a+b\sin x}=\frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}+C,\text{ when } a<b.$$

162. $$\int\frac{dx}{a^2\cos^2x+b^2\sin^2x}=\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)+C.$$

163. $$\int e^{ax}\sin nx\,dx=\frac{e^{ax}(a\sin nx-n\cos nx)}{a^2+n^2}+C;$$

$$\int e^x\sin x\,dx=\frac{e^x(\sin x-\cos x)}{2}+C.$$

164. $\int e^{ax}\cos nx\,dx = \dfrac{e^{ax}(n\sin nx + a\cos nx)}{a^2+n^2} + C;$

$\int e^{x}\cos x\,dx = \dfrac{e^{x}(\sin x + \cos x)}{2} + C.$

165. $\int xe^{ax}dx = \dfrac{e^{ax}}{a^2}(ax-1) + C.$

166. $\int x^n e^{ax}dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}dx.$

167. $\int a^{mx}x^n dx = \dfrac{a^{mx}x^n}{m\log a} - \dfrac{n}{m\log a}\int a^{mx}x^{n-1}dx.$

168. $\int \dfrac{a^x dx}{x^m} = -\dfrac{a^x}{(m-1)x^{m-1}} + \dfrac{\log a}{m-1}\int\dfrac{a^x dx}{x^{m-1}}.$

169. $\int e^{ax}\cos^n x\,dx = \dfrac{e^{ax}\cos^{n-1}x\,(a\cos x + n\sin x)}{a^2+n^2} + \dfrac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx.$

170. $\int x^m\cos ax\,dx = \dfrac{x^{m-1}}{a^2}(ax\sin ax + m\cos ax) - \dfrac{m(m-1)}{a^2}\int x^{m-2}\cos ax\,dx.$

### LOGARITHMIC FORMS

171. $\int \log x\,dx = x\log x - x + C.$

172. $\int \dfrac{dx}{\log x} = \log(\log x) + \log x + \dfrac{1}{2^2}\log^2 x + \cdots,$

173. $\int \dfrac{dx}{x\log x} = \log(\log x) + C.$

174. $\int x^n\log x\,dx = x^{n+1}\left[\dfrac{\log x}{n+1} - \dfrac{1}{(n+1)^2}\right] + C.$

175. $\int e^{ax}\log x\,dx = \dfrac{e^{ax}\log x}{a} - \dfrac{1}{a}\int\dfrac{e^{ax}}{x}dx.$

176. $\int x^m\log^n x\,dx = \dfrac{x^{m+1}}{m+1}\log^n x - \dfrac{n}{m+1}\int x^m\log^{n-1}x\,dx.$

177. $\int \dfrac{x^m dx}{\log^n x} = -\dfrac{x^{m+1}}{(n-1)\log^{n-1}x} + \dfrac{m+1}{n-1}\int\dfrac{x^m dx}{\log^{n-1}x}.$

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